On the average principle for one-frequency systems.

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Abstract

We consider a perturbed integrable system with one frequency, and the approximate dynamics for the actions given by averaging over the angle. A classical qualitative result states that, for a perturbation of order ε, the error of this approximation is $O(\varepsilon)$ on a time scale $O(1/\varepsilon)$, for $\varepsilon \to 0$. We replace this with a fully quantitative estimate; in certain cases, our approach also gives a reliable error estimate on time scales larger than $1/\varepsilon$. A number of examples are presented; in many cases, our estimator practically coincides with the envelope of the rapidly oscillating distance between the actions of the perturbed and of the averaged systems. Fairly good results are also obtained in some "resonant" cases, where the angular frequency is small along the trajectory of the system.

Even though our estimates are proved theoretically, their computation in specific applications typically requires the numerical solution of a system of differential equations. However, the time scale for this system is smaller by a factor ε than the time scale for the perturbed system. For this reason, computation of our estimator is faster than the direct numerical solution of the perturbed system; the estimator is rapidly found also in cases when the time scale makes impossible (within reasonable CPU times) or unreliable the direct solution of the perturbed system.

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1 Introduction.

The averaging method is a classical tool to analyse dynamical systems with fast angular variables: the idea is to average over the angles, to obtain an approximate evolution law for the slow variables (from now on, called the actions). Many applications are physically relevant; so, error estimates for this technique on long time scales have an obvious interest.
Concerning these estimates, the case of one angle is the simplest one due to the structure of its "resonances", which are produced only by the vanishing of the angular frequency. However, this one-frequency case covers non trivial situations: for example, it includes the perturbed Kepler problem, appearing in applications such as the dynamics of a satellite around an oblate planet and/or in presence of dragging (see [7] and references therein).
The classical theory for the one-frequency case states that, under a perturbation $O(\varepsilon)$ of a dynamical system with one angle and many actions, the difference between the actions of the perturbed and of the averaged systems is $O(\varepsilon)$ on a time scale $O(1/\varepsilon)$, for $\varepsilon \to 0$: see [1] [2] [3] [4] [7] (the two last references are also useful for general historical and bibliographical information). This is a qualitative result; the n-th order extensions of the averaging method proposed in the literature [4] are usually treated at the same qualitative level, the conclusion being that some reminder term is $O(\varepsilon^n)$ on a time scale $O(1/\varepsilon)$. To get these $O(\varepsilon)$ or $O(\varepsilon^n)$ bounds, one generally writes a number of quite rough majorizations, often containing unspecified constants but sufficient to obtain a linear integral inequality for the reminder; the latter is used to obtain the wanted bounds through the Gronwall Lemma.
Of course, the previously mentioned results are not fully satisfactory if one aims to obtain precise numerical values from the error analysis; the situation is especially uncomfortable near resonances, i.e., when the time evolution carries the system close to a zero of the angular frequency.
In this paper we show that working carefully, and avoiding unnecessary simplifications, it is possible to derive fully quantitative and precise error estimates for the standard ($n = 1$) averaging method, for a (small) fixed ε: this requires to solve a nonlinear integral inequality, or a related differential equation, coupled to a set of auxiliary differential equations. In typical cases, this is done numerically; however, the treatment of the above system of equations is much less expensive than the direct numerical solution of the action-angle evolution equations; in fact, to get information on an interval $[0, U/\varepsilon)$ it suffices to solve the previously mentioned set of equations on the interval $[0, U)$.
To our knowledge, a quantitative error analysis for the averaging method has been previously proposed in [8]; however, in this reference the attention is mainly focused on specific applications, admitting a simple analytical treatment, rather than on a general scheme. In a broader sense, the present paper has some connection with [6]; in the cited reference, a quantitative analysis has been proposed for a rather general

class of approximation methods for the evolution equations (in abstract Banach spaces, so to include the case of evolutionary PDEs).

1A. A precise setting of the problem. Let us be given an open set Λ of $\mathbf{R}^d$ (the space of the actions) and the one-dimensional torus $\mathbf{T}$:

$$\Lambda = \{I = (I^i)_{i=1,\dots,d}\} \subset \mathbf{R}^d \ , \qquad \mathbf{T} := \mathbf{R}/(2\pi\mathbf{Z}) = \{\vartheta\} \ . \tag{1.1}$$

We fix some initial data

$$I_0 \in \Lambda \ , \qquad \vartheta_0 \in \mathbf{T} \tag{1.2}$$

and consider the perturbed one-frequency system

$$\begin{cases} d\mathtt{I}/dt = \varepsilon f(\mathtt{I}, \Theta) \ , & \mathtt{I}(0) = I_0 \ , \\ d\Theta/dt = \omega(\mathtt{I}) + \varepsilon g(\mathtt{I}, \Theta) \ , & \Theta(0) = \vartheta_0 \end{cases} \tag{1.3}$$

for two unknown functions $\mathtt{I} : t \mapsto \mathtt{I}(t) \in \Lambda$, $\Theta : t \mapsto \Theta(t) \in \mathbf{T}$. This Cauchy problem contains the unperturbed frequency

$$\omega \in C^m(\Lambda, \mathbf{R}) \ , \qquad \omega(I) \neq 0 \text{ for all } I \in \Lambda \ ; \tag{1.4}$$

the perturbation is governed by a parameter $\varepsilon > 0$, and by two functions

$$f = (f^i)_{i=1,\dots,d} \in C^m(\Lambda \times \mathbf{T}, \mathbf{R}^d), \qquad g \in C^m(\Lambda \times \mathbf{T}, \mathbf{R}), \tag{1.5}$$
$$(I, \vartheta) \mapsto f(I, \vartheta), \ g(I, \vartheta) \ ;$$

throughout the paper, for technical reasons it is assumed that $m \geqslant 2$.
From now on "the solution $(\mathtt{I}, \Theta)$ of (1.3)" means the *maximal* solution *in the future*, i.e., the one with the largest domain of the form $[0, T)$, $T \in (0, +\infty]$ (of course, this domain generally depends on the initial data). Any expression like "the solution $(\mathtt{I}, \Theta)$ exists on D" means that D is a subset of $[0, T)$. It is hardly the case to observe that $\mathtt{I}, \Theta$ are C^{m+1} functions.
The averaged system associated to (1.3) is the Cauchy problem

$$\frac{d\mathtt{J}}{d\tau} = \overline{f}(\mathtt{J}) \ , \qquad \mathtt{J}(0) = I_0 \ , \tag{1.6}$$

$$\overline{f} = (\overline{f^i}) \in C^m(\Lambda, \mathbf{R}^d) \ , \qquad I \mapsto \overline{f}(I) := \frac{1}{2\pi} \int_{\mathbf{T}} d\vartheta \ f(I, \vartheta) \ ;$$

the unknown is a function $\mathtt{J} : \tau \mapsto \mathtt{J}(\tau) \in \Lambda$. In the same language as before, we stipulate that "the solution $\mathtt{J}$ of (1.6)" means the maximal one in the future; again, we have a C^{m+1} function.
The system (1.6) will be compared with (1.3) for $\tau = \varepsilon t$, i.e., interpreting τ as a rescaled time; if $(\mathtt{I}, \Theta)$ is the solution of (1.3) and $\mathtt{J}$ is the solution of (1.6) *with the same datum* I_0 *as in* (1.3), the aim is to evaluate the difference $t \mapsto \mathtt{I}(t) - \mathtt{J}(\varepsilon t)$.

The classical result on this subject is an estimate

$$|\mathtt{I}(t) - \mathtt{J}(\varepsilon\, t)| \leqslant C\varepsilon \qquad \text{for } t \in [0, 1/\varepsilon) \ , \tag{1.7}$$

holding for all sufficiently small ε, under suitable technical conditions (especially, a lower bound $|\omega(I)| \geqslant c > 0$ on a convenient domain); in the above, C is a constant independent of ε. In principle, one could obtain for C a (very complicated) expression, for example evaluating all the constants in the derivation of (1.7) by [1]; however, the explicit bound obtained in this way is not satisfactory, since in typical examples it largely overestimates the difference $\mathtt{I}(t) - \mathtt{J}(\varepsilon t)$.

1B. Contents of the paper. Throughout the paper, the parameter ε is *fixed* in $(0, +\infty)$; of course, our statements are interesting mainly if ε is small (and are accompanied by comments which assume this). Our aim is to perform an accurate analysis of the distance between $\mathtt{I}$ and $\mathtt{J}$; this will ultimately yield a bound

$$|\mathtt{I}(t) - \mathtt{J}(\varepsilon t)| \leqslant \varepsilon\, \mathfrak{n}(\varepsilon t) \qquad \text{for } t \in [0, U/\varepsilon) \ , \tag{1.8}$$

where $\mathfrak{n} : \tau \mapsto \mathfrak{n}(\tau)$ fulfils an integral inequality, or a related differential equation, for τ within an interval $[0, U)$. (As we will show, the existence of $\mathtt{J}, \mathfrak{n}$ and some more auxiliary functions for $\tau \in [0, U)$ grants the existence of the solution $(\mathtt{I}, \Theta)$ of (1.3) for $t \in [0, U/\varepsilon)$).

Typically, the estimator $\mathfrak{n}$ must be computed solving numerically the above mentioned differential equation; however, this is much less expensive than the numerical solution of (1.3), because $\mathfrak{n}$ depends on the "slow" time variable $\tau = \varepsilon t$ and thus must be determined on an interval of length U to get an estimate for $t \in [0, U/\varepsilon)$ (these considerations can be extended to all the auxiliary functions required in this approach). In the examples we will present, the function $t \mapsto \varepsilon \mathfrak{n}(\varepsilon t)$ obtained in this way often coincides with the "envelope" of the rapidly oscillating function $t \mapsto |\mathtt{I}(t) - \mathtt{J}(\varepsilon t)|$, giving practically the best possible bound of the form (1.8). Our bound turns out to be fairly good also in some resonant cases (where ω vanishes at the boundary of Λ and the actions are close to it, either initially or over long times). As expected, in each example the CPU time for the computation of $\mathfrak{n}$ is much shorter than the CPU time for the direct solution of (1.3).

If $U \simeq 1$, Eq. (1.8) can be regarded as a quantitative formulation of the classical theory, involving the time scales 1 and $1/\varepsilon$. However, in certain cases our approach works as well for $U \gg 1$, yielding accurate estimates for $|\mathtt{I}(t) - \mathtt{J}(\varepsilon t)|$ on the extremely large interval $[0, U/\varepsilon)$; one can even jump to the time scales $U \simeq 1/\varepsilon$, $U/\varepsilon \simeq 1/\varepsilon^2$.

The general setting of our approach is described in Section 2, where we use systematically the function

$$t \mapsto \mathtt{L}(t) := \frac{1}{\varepsilon}[\mathtt{I}(t) - \mathtt{J}(\varepsilon t)] \ . \tag{1.9}$$

After introducing a set of auxiliary functions and differential equations, in Lemma 2.1 we obtain an exact integral equation for $\mathtt{L}$; then, in Proposition 2.4 we derive an

integral inequality and show that any solution $\tau \mapsto \mathfrak{n}(\tau)$ of this inequality gives a bound $|\mathtt{L}(t)| < \mathfrak{n}(\varepsilon t)$. For practical purposes, it is convenient to relate the integral inequality for $\mathfrak{n}$ to a differential equation, which is the subject of Proposition 2.5; the solution $\mathfrak{n}$ of the differential equation gives a bound $|\mathtt{L}(t)| \leqslant \mathfrak{n}(\varepsilon t)$, which is equivalent to Eq.(1.8).

The subsequent Section 3 summarizes the path to $\mathfrak{n}$, and discusses tests for the efficiency of this estimator. The final Section 4 is devoted to the examples: we mention, in particular, the van der Pol equation, a resonant case inspired by Arnold, and Euler's equations for a rigid body under a damping moment linear in the angular velocity (which also manifest a resonance).

To simplify our exposition, many technical aspects are treated in the Appendices. In particular: Appendices A, B and C contain the proofs of Lemmas 2.1, 2.3 and Proposition 2.5, respectively; Appendices D and E illustrate the computation of some auxiliary functions required by the examples of Section 4.

The examples presented in this paper are relatively simple, since their purpose is mainly to test the effectiveness of the method. We postpone to later works (now in progress) the treatment of slightly harder applications, in particular the already mentioned satellite dynamics.

2 Main results.

2A. Some notations. i) Vectors of $\mathbf{R}^d$ are written with upper indices: $X = (X^i)_{i=1,...,d}$. We use the spaces $\mathrm{T}^p_q(\mathbf{R}^d)$ of (p,q)-tensors over $\mathbf{R}^d$, especially for $(p,q) = (1,1), (2,0)$ and $(1,2)$; tensors of these three types are represented as families of real coefficients $\mathscr{A} = (\mathscr{A}^i_j)$, $\mathscr{B} = (\mathscr{B}^{ij})$, $\mathscr{C} = (\mathscr{C}^i_{jk})$ $(i,j,k = 1,...,d)$.

Let $X, Y \in \mathbf{R}^d$, $\mathscr{A}, \mathscr{D} \in \mathrm{T}^1_1(\mathbf{R}^d)$, $\mathscr{B} \in \mathrm{T}^2_0(\mathbf{R}^d)$, $\mathscr{C} \in \mathrm{T}^1_2(\mathbf{R}^d)$. We define the products $XY \in \mathrm{T}^2_0(\mathbf{R}^d)$, $\mathscr{A}X \in \mathbf{R}^d$, $\mathscr{A}\mathscr{D} \in \mathrm{T}^1_1(\mathbf{R}^d)$, $\mathscr{C}X \in \mathrm{T}^1_1(\mathbf{R}^d)$, $\mathscr{C}\mathscr{B} \in \mathbf{R}^d$ by

$$(XY)^{ij} := X^i Y^j \ , \qquad (\mathscr{A}X)^i := \mathscr{A}^i_k X^k \ , \qquad (\mathscr{A}\mathscr{D})^i_j := \mathscr{A}^i_k \mathscr{D}^k_j \ , \tag{2.1}$$
$$(\mathscr{C}X)^i_\ell = \mathscr{C}^i_{k\ell} X^k \ , \qquad (\mathscr{C}\mathscr{B})^i := \mathscr{C}^i_{k\ell} \mathscr{B}^{k\ell}$$

(with the Einstein's summation convention over repeated indices; XX will be written X^2). We note that $\mathscr{A}\mathscr{D}$ is the ordinary product of $\mathscr{A}$ and $\mathscr{D}$ as matrices; 1_d, $\mathscr{A}^{-1}$ $\in \mathrm{T}^1_1(\mathbf{R}^d)$ will denote the identity matrix, and the inverse matrix of $\mathscr{A}$. The vector $(\mathscr{C}X)Y = \mathscr{C}(XY)$ will be written $\mathscr{C}XY$.

All the considered tensor spaces can be equipped with an inner product $\bullet$ and with the corresponding Euclidean norm $|\ |$. If $X, Y \in \mathbf{R}^d$, $\mathscr{A}, \mathscr{D} \in \mathrm{T}^1_1(\mathbf{R}^d)$ and $\mathscr{C}, \mathscr{E} \in \mathrm{T}^1_2(\mathbf{R}^d)$,

$$X \bullet Y := \sum_{i=1}^d X^i Y^i \ , \qquad \mathscr{A} \bullet \mathscr{D} = \sum_{i,j=1}^d \mathscr{A}^i_j \, \mathscr{D}^i_j \ , \qquad \mathscr{C} \bullet \mathscr{E} := \sum_{i,j,k=1}^d \mathscr{C}^i_{jk} \, \mathscr{E}^i_{jk} \ , \tag{2.2}$$

$$|X| := \sqrt{X \bullet X}\ , \qquad |\mathscr{A}| := \sqrt{\mathscr{A} \bullet \mathscr{A}}\ , \qquad |\mathscr{C}| := \sqrt{\mathscr{C} \bullet \mathscr{C}}\ .$$

ii) Recalling that $\Lambda \subset \mathbf{R}^d$ is open, let $h : \Lambda \to \mathbf{R}^d$ be C^ℓ. If $\ell \geqslant 1$ or $\ell \geqslant 2$, respectively, the Jacobian and the Hessian of h at a point I are

$$\frac{\partial h}{\partial I}(I) := \left(\frac{\partial h^i}{\partial I^j}(I)\right) \in \mathrm{T}^1_1(\mathbf{R}^d)\ ; \qquad \frac{\partial^2 h}{\partial I^2}(I) := \left(\frac{\partial^2 h^i}{\partial I^j \partial I^k}(I)\right) \in \mathrm{T}^1_2(\mathbf{R}^d)\ . \quad (2.3)$$

Let us introduce the set (open in $\mathbf{R}^d \times \mathbf{R}^d$).

$$\Lambda_\dagger := \{(I, \delta I) \in \Lambda \times \mathbf{R}^d \mid [I, I + \delta I] \in \Lambda\} \quad (2.4)$$

(with $[I, I+\delta I]$ denoting the segment of $\mathbf{R}^d$ with the indicated extremes. For h as before and $\ell \geqslant 1$ or $\ell \geqslant 2$, respectively, there are functions $\mathscr{G} \in C^{\ell-1}(\Lambda_\dagger, \mathrm{T}^1_1(\mathbf{R}^d))$ and $\mathscr{H} \in C^{\ell-2}(\Lambda_\dagger, \mathrm{T}^1_2(\mathbf{R}^d))$ such that

$$h(I + \delta I) = h(I) + \mathscr{G}(I, \delta I)\delta I\ , \quad (2.5)$$

$$h(I + \delta I) = h(I) + \frac{\partial h}{\partial I}(I)\delta I + \frac{1}{2}\mathscr{H}(I, \delta I)\delta I^2\ , \qquad \mathscr{H}^i_{jk}(I, \delta I) = \mathscr{H}^i_{kj}(I, \delta I)\ . \quad (2.6)$$

If $d = 1$, the above equations can be solved for $\mathscr{G}$, $\mathscr{H}$ and determine them uniquely. If $d > 1$, the above equations for $\mathscr{G}$, $\mathscr{H}$ have many solutions; in any dimension, explicit solutions are given by the integral formulas

$$\mathscr{G}(I, \delta I) := \int_0^1 dx \frac{\partial h}{\partial I}(I + x\delta I)\ , \quad \mathscr{H}(I, \delta I) := 2\int_0^1 dx (1-x)\frac{\partial^2 h}{\partial I^2}(I + x\delta I) \quad (2.7)$$

(for h of polynomial or rational type, $\mathscr{G}$ and $\mathscr{H}$ can be obtained more directly from the expression of $h(I + \delta I)$).
In an obvious way, for a function $h : \Lambda \times \mathbf{T} \to \mathbf{R}^d$, we can define the derivatives $(\partial h/\partial I)(I, \vartheta) \in \mathrm{T}^1_1(\mathbf{R}^d)$, $(\partial h/\partial \vartheta)(I, \vartheta) \in \mathbf{R}^d$, $(\partial^2 h/\partial I^2)(I, \vartheta) \in \mathrm{T}^1_2(\mathbf{R}^d)$.

iii) The average of a C^ℓ function $h : \Lambda \times \mathbf{T} \to \mathbf{R}^d$ is the C^ℓ function $\overline{h} : \Lambda \to \mathbf{R}^d$, $I \mapsto \overline{h}(I) := 1/(2\pi)\int_{\mathbf{T}} d\vartheta\, h(I, \vartheta)$; this notation has been already used in Eq. (1.6), with $h = f$.

2B. The main Lemma: an integral equation for $\mathtt{L}$. We consider the perturbed and averaged systems (1.3) (1.6), for fixed $\varepsilon > 0$ and initial data I_0, ϑ_0.
The integral equation we are going to derive will be the basic identity yielding our estimates on $|\mathtt{L}(t)|$; it involves a number of auxiliary functions, to be introduced as the construction goes on.
First of all, $s \in C^m(\Lambda \times \mathbf{T}, \mathbf{R}^d)$ and $p \in C^{m-1}(\Lambda \times \mathbf{T}, \mathbf{R}^d)$ are the functions such that

$$f = \overline{f} + \omega\, \frac{\partial s}{\partial \vartheta}\ , \quad \overline{s} = 0\ ; \qquad p := \frac{\partial s}{\partial I} f + \frac{\partial s}{\partial \vartheta} g\ . \quad (2.8)$$

The function s, which has a preminent role in estimates on $|\mathtt{L}|$, is defined by (2.8) in an implicit way; an explicit formula is ([1])

$$s = z - \overline{z} \ , \qquad z(I,\vartheta) := \frac{1}{\omega(I)} \int_0^{\vartheta} d\vartheta' \ (f(I,\vartheta') - \overline{f}(I)) \ . \tag{2.9}$$

Another function to be used hereafter is the Jacobian $\dfrac{\partial \overline{f}}{\partial I} \in C^{m-1}(\Lambda, \mathrm{T}^1_1(\mathbf{R}^d))$. From now on, U stands for an element of $(0,+\infty]$.

2.1 *Lemma.* *Suppose the solution* $\mathtt{J}$ *of (1.6) exists for* $\tau \in [0,U)$. *Denote with* $\mathtt{R} : [0,U) \to T^1_1(\boldsymbol{R}^d)$, $\tau \mapsto \mathtt{R}(\tau)$ *and* $\mathtt{K} : [0,U) \to \boldsymbol{R}^d$, $\tau \mapsto \mathtt{K}(\tau)$ *the solutions of*

$$\frac{d\mathtt{R}}{d\tau} = \frac{\partial \overline{f}}{\partial I}(\mathtt{J})\,\mathtt{R} \ , \qquad \mathtt{R}(0) = 1_d \ ; \tag{2.10}$$

$$\frac{d\mathtt{K}}{d\tau} = \frac{\partial \overline{f}}{\partial I}(\mathtt{J})\,\mathtt{K} + \overline{p}(\mathtt{J}) \ , \qquad \mathtt{K}(0) = 0 \tag{2.11}$$

(these exist and are C^m*;* $\mathtt{R}(\tau)$ *is an invertible matrix for all* $\tau \in [0,U)$*, and* $\mathtt{K}(\tau) = \mathtt{R}(\tau)\int_0^\tau d\tau'\, \mathtt{R}(\tau')^{-1}\overline{p}(\mathtt{J}(\tau'))$*. For* $d=1$*,* $\mathtt{R}(\tau) = \exp\int_0^\tau d\tau' \dfrac{\partial \overline{f}}{\partial I}(\mathtt{J}(\tau')) \in (0,+\infty)$*). Furthermore, assume that the solution* $(\mathtt{I},\Theta)$ *of the perturbed system (1.3) exixts for* $t \in [0, U/\varepsilon)$*, with* $(\mathtt{J}(\varepsilon t), \mathtt{I}(t) - \mathtt{J}(\varepsilon t)) \in \Lambda_\dagger$*. Finally, define*

$$\mathtt{L} : [0, U/\varepsilon) \to \boldsymbol{R}^d \ , \qquad t \mapsto \mathtt{L}(t) := \frac{1}{\varepsilon}\left[\mathtt{I}(t) - \mathtt{J}(\varepsilon t)\right] \ . \tag{2.12}$$

Then, for $t \in [0, U/\varepsilon)$ *it is*

$$\begin{aligned} \mathtt{L}(t) &= s(\mathtt{I}(t),\Theta(t)) - \mathtt{R}(\varepsilon t)\, s(I_0,\vartheta_0) - \mathtt{K}(\varepsilon t) + \\ &\quad -\varepsilon\Big(w(\mathtt{I}(t),\Theta(t)) - \frac{\partial \overline{f}}{\partial I}(\mathtt{J}(\varepsilon t))\, v(\mathtt{I}(t),\Theta(t))\Big) \ + \\ &\quad + \varepsilon^2 \mathtt{R}(\varepsilon t) \int_0^t dt'\, \mathtt{R}^{-1}(\varepsilon t') \Big(u(\mathtt{I}(t'),\Theta(t')) - \frac{\partial \overline{f}}{\partial I}(\mathtt{J}(\varepsilon t'))(w+q)(\mathtt{I}(t'),\Theta(t')) + \\ &\quad - \mathscr{M}(\mathtt{J}(\varepsilon t')) v(\mathtt{I}(t'),\Theta(t')) - \mathscr{G}(\mathtt{J}(\varepsilon t'), \varepsilon \mathtt{L}(t'))\mathtt{L}(t') + \frac{1}{2}\mathscr{H}(\mathtt{J}(\varepsilon t'), \varepsilon\mathtt{L}(t'))\,\mathtt{L}(t')^2 \Big) \ . \end{aligned} \tag{2.13}$$

In the above, $v \in C^m(\Lambda \times \boldsymbol{T}, \boldsymbol{R}^d)$*,* $q, w \in C^{m-1}(\Lambda \times \boldsymbol{T}, \boldsymbol{R}^d)$*,* $u \in C^{m-2}(\Lambda \times \boldsymbol{T}, \boldsymbol{R}^d)$*, and* $\mathscr{M} \in C^{m-2}(\Lambda, T^1_1(\boldsymbol{R}^d))$ *are the functions uniquely defined by the following equations:*

$$s = \omega \frac{\partial v}{\partial \vartheta} \ , \qquad v(I,\vartheta_0) = 0 \quad \text{for all } I \in \Lambda; \tag{2.14}$$

[1] here, $\int_0^\vartheta$ means integration along any path in $\mathbf{T}$ from 0 to ϑ; the integral depends only on the extremes, because the integrand has zero average. The same could be said for other integrals appearing later.

$$q := \frac{\partial v}{\partial I} f + \frac{\partial v}{\partial \vartheta} g \ ; \tag{2.15}$$

$$p = \overline{p} + \omega \frac{\partial w}{\partial \vartheta} \ , \qquad w(I, \vartheta_0) = 0 \quad \textit{for all } I \in \Lambda \ ; \tag{2.16}$$

$$u := \frac{\partial w}{\partial I} f + \frac{\partial w}{\partial \vartheta} g \ ; \qquad \mathscr{M} := \frac{\partial^2 \overline{f}}{\partial I^2} \overline{f} - \left(\frac{\partial \overline{f}}{\partial I} \right)^2 \ . \tag{2.17}$$

Furthermore, $\mathscr{G} \in C^{m-2}(\Lambda_\dagger, T^1_1(\boldsymbol{R}^d))$ *and* $\mathscr{H} \in C^{m-2}(\Lambda_\dagger, T^1_2(\boldsymbol{R}^d))$ *are two functions fulfilling Eq.s (2.5) for* $h = \overline{p}$ *and (2.6) for* $h = \overline{f}$*: so, for* $(I, \delta I) \in \Lambda_\dagger$*,*

$$\overline{p}(I + \delta I) = \overline{p}(I) + \mathscr{G}(I, \delta I)\delta I \ , \tag{2.18}$$

$$\overline{f}(I + \delta I) = \overline{f}(I) + \frac{\partial \overline{f}}{\partial I}(I)\delta I + \frac{1}{2}\mathscr{H}(I, \delta I)\delta I^2 \ , \quad \mathscr{H}^i_{jk}(I, \delta I) = \mathscr{H}^i_{kj}(I, \delta I) \ . \tag{2.19}$$

Proof. It is obtained by a long computation, where the functions $s, ..., \mathscr{H}$ appear gradually. See the Appendix A. $\diamond$

Remarks. i) The above definitions must be understood in terms of the previous tensor notations; for example, the equivalent formulation in components of Eq. (2.17) is $\mathscr{M}^i_j = \dfrac{\partial^2 \overline{f^i}}{\partial I^k \partial I^j} \overline{f^k} - \dfrac{\partial \overline{f^i}}{\partial I^k} \dfrac{\partial \overline{f^k}}{\partial I^j}$. Of course $v(I, \vartheta) = \omega^{-1}(I) \int_{\vartheta_0}^{\vartheta} d\vartheta' \ s(I, \vartheta')$, $w(I, \vartheta) = \omega^{-1}(I)) \int_0^{\vartheta} d\vartheta' \ (p(I, \vartheta') - \overline{p}(I))$.
ii) If we write $\mathtt{I}(t) = \mathtt{J}(\varepsilon t) + \varepsilon \mathtt{L}(t)$, (2.13) becomes an integral equation for $\mathtt{L}$. Most of the terms therein are slow, i.e., depend on εt: the exceptions are $\mathtt{L}$ itself and the angle Θ. The subsequent step after this Lemma will be to infer from (2.13) an integral inequality involving only the slow time variable εt; we note that, even though the integral in Eq. (2.13) is multiplied by ε^2, this term appears to be of order ε if we consider $\varepsilon t'$ as the integration variable. In any case, the presence of a small factor ε in front of the integral allows us to use for it fairly rough estimates.

2C. A second Lemma: an integral inequality for $|\mathtt{L}|$. Throughout this paragraph we assume that the solution $\mathtt{J}$ of the averaged system exists on $[0, U)$, and define $\mathtt{R}, \mathtt{K}$ via Eq.s (2.10), (2.11). $B(I, \varrho)$ denotes the open ball in $\mathbf{R}^d$ of center I and radius ϱ; we furtherly suppose the following.
i) There is a function $\rho \in C([0, U), [0, +\infty])$ such that

$$B(\mathtt{J}(\tau), \rho(\tau)) \subset \Lambda \qquad \text{for } \tau \in [0, U) \ . \tag{2.20}$$

We denote with Γ_ρ the subgraph of ρ, i.e.,

$$\Gamma_\rho := \{ (\tau, r) \mid \tau \in [0, U), \ r \in [0, \rho(\tau)) \ \} \ . \tag{2.21}$$

ii) There are functions

$$a, b, c, d, e \in C(\Gamma_\rho, [0, +\infty)) \tag{2.22}$$

such that, for any $\tau \in [0, U)$, $\delta J \in B(0, \rho(\tau))$ and $\vartheta \in \mathbf{T}$,

$$|s(\mathtt{J}(\tau) + \delta J, \vartheta) - \mathtt{R}(\tau) s(I_0, \vartheta_0) - \mathtt{K}(\tau)| \leqslant a(\tau, |\delta J|) \ , \tag{2.23}$$

$$\left| w(\mathtt{J}(\tau) + \delta J, \vartheta) - \frac{\partial \overline{f}}{\partial I}(\mathtt{J}(\tau))\, v(\mathtt{J}(\tau) + \delta J, \vartheta) \right| \leqslant b(\tau, |\delta J|) \ , \tag{2.24}$$

$$\begin{aligned} &\Big| u(\mathtt{J}(\tau) + \delta J, \vartheta) - \frac{\partial \overline{f}}{\partial I}(\mathtt{J}(\tau))(w + q)(\mathtt{J}(\tau) + \delta J, \vartheta) + \\ &\qquad - \mathscr{M}(\mathtt{J}(\tau)) v(\mathtt{J}(\tau) + \delta J, \vartheta) \Big| \leqslant c(\tau, |\delta J|) \ , \end{aligned} \tag{2.25}$$

$$|\mathscr{G}(\mathtt{J}(\tau), \delta J)| \leqslant d(\tau, |\delta J|) \ , \tag{2.26}$$

$$|\mathscr{H}(\mathtt{J}(\tau), \delta J)| \leqslant e(\tau, |\delta J|) \tag{2.27}$$

(note that $(\mathtt{J}(\tau), \delta J) \in \Lambda_\dagger$, by the convexity of the sphere). The functions c, d, e are assumed to be non decreasing with respect to the second variable:

$$(\tau, r), (\tau, r') \in \Gamma_\rho, \quad r \leqslant r' \quad \Rightarrow \quad c(\tau, r) \leqslant c(\tau, r') \ , \tag{2.28}$$

and similarly for d, e. Given a, b, c, d, e, we define the functions

$$\alpha \in C(\Gamma_\rho, [0, +\infty)), \quad \alpha(\tau, r) := a(\tau, r) + \varepsilon b(\tau, r) \ , \tag{2.29}$$

$$\gamma \in C(\Gamma_\rho \times [0, +\infty), [0, +\infty)), \quad \gamma(\tau, r, \ell) := c(\tau, r) + d(\tau, r)\ell + \frac{1}{2} e(\tau, r)\ell^2 \ . \tag{2.30}$$

We can now write the integral inequality for the function $t \mapsto |\mathtt{L}(t)|$, with $\mathtt{L}$ as in (2.12).

2.2 ***Lemma*** *Assume that the solution $(\mathtt{I}, \Theta)$ of the perturbed system exists on $[0, U/\varepsilon)$, and that $|\mathtt{L}(t)| < \rho(\varepsilon t)/\varepsilon$ for all $t \in [0, U/\varepsilon)$. Then*

$$|\mathtt{L}(t)| \leqslant \alpha(\varepsilon t, \varepsilon |\mathtt{L}(t)|) + \varepsilon^2 |\mathtt{R}(\varepsilon t)| \int_0^t dt' \, |\mathtt{R}^{-1}(\varepsilon t')| \, \gamma(\varepsilon t', \varepsilon |\mathtt{L}(t')|, |\mathtt{L}(t')|) \ . \tag{2.31}$$

Proof. We take the norm of both sides in Eq. (2.13). To estimate the right hand side, we use some Schwarz inequalities and Eq.s (2.23–2.27) with $\delta J = \mathtt{I}(t) - \mathtt{J}(\varepsilon t) = \varepsilon \mathtt{L}(t)$; then, the thesis follows from the definitions (2.29) (2.30) of α and γ. $\diamond$

2D. A third Lemma, on integral inequalities. To go on, we need a general result on a class of integral inequalities; we state it at an abstract level, forgetting momentarily the function $|\mathtt{L}|$.

2.3 *Lemma.* *Let* $T \in (0, +\infty]$, $\delta \in C([0, T), [0, +\infty])$ *and*

$$\Xi := \{(t, \ell) \mid t \in [0, T), \ell \in [0, \delta(t))\ \}, \tag{2.32}$$

$$\mathrm{H} := \{(t, t', \ell) \mid t \in [0, T)\ , t' \in [0, t],\ (t', \ell) \in \Xi\ \}\ .$$

Consider two functions $\xi \in C(\Xi, [0, +\infty))$ *and* $\eta \in C(\mathrm{H}, [0, +\infty))$, *the latter non decreasing in the last variable:* $\eta(t, t', \ell') \leqslant \eta(t, t', \ell)$ *for* $(t, t', \ell) \in \mathrm{H}$ *and* $\ell' \in [0, \ell]$. *Furthermore, let* $\mathfrak{l} \in C([0, T), [0, +\infty))$ *and* $\mathfrak{v} \in C([0, T), (0, +\infty))$ *be such that graph* $\mathfrak{l}$, *graph* $\mathfrak{v} \subset \Xi$, *and*

$$\mathfrak{l}(0) = 0\ , \qquad \mathfrak{l}(t) \leqslant \xi(t, \mathfrak{l}(t)) + \int_0^t dt'\, \eta(t, t', \mathfrak{l}(t'))\ , \tag{2.33}$$

$$\mathfrak{v}(t) > \xi(t, \mathfrak{v}(t)) + \int_0^t dt'\, \eta(t, t', \mathfrak{v}(t')) \tag{2.34}$$

for all $t \in [0, T)$. *Then*

$$\mathfrak{l}(t) < \mathfrak{v}(t) \qquad \textit{for all } t \in [0, T)\ . \tag{2.35}$$

Proof. It adapts the one of a similar result in [5]; see the Appendix B. $\diamond$

2E. The main Proposition. Throughout this paragraph we still assume that the solution $\mathtt{J}$ of the averaged system exists on $[0, U)$, and define $\mathtt{R}, \mathtt{K}$ via Eq.s (2.10) (2.11). We also assume there is a set of functions ρ, a, b, c, d, e as in paragraph 2C; α and γ are defined consequently, as indicated therein.

2.4 *Proposition.* *Assume that there is a function* $\mathfrak{n} \in C([0, U), (0, +\infty))$ *such that, for all* $\tau \in [0, U)$,

$$\mathfrak{n}(\tau) < \rho(\tau)/\varepsilon\ , \tag{2.36}$$

$$\mathfrak{n}(\tau) > \alpha(\tau, \varepsilon\mathfrak{n}(\tau)) + \varepsilon|\mathtt{R}(\tau)| \int_0^\tau d\tau'\, |\mathtt{R}^{-1}(\tau')|\, \gamma(\tau', \varepsilon\mathfrak{n}(\tau'), \mathfrak{n}(\tau'))\ . \tag{2.37}$$

Then, the solution $(\mathtt{I}, \Theta)$ *of the perturbed system exists on* $[0, U/\varepsilon)$; *furthermore, defining* $\mathtt{L}$ *as in Eq. (2.12) we have*

$$|\mathtt{L}(t)| < \mathfrak{n}(\varepsilon t) \qquad \textit{for all } t \in [0, U/\varepsilon). \tag{2.38}$$

Proof. Let us recall that $(\mathtt{I}, \Theta)$ is the *maximal* solution of (1.3), and denote its domain with $[0, V/\varepsilon)$; for the moment, this merely defines the coefficient $V \in (0, +\infty]$ (which can depend on ε and be large, small, etc.). To go on, we provisionally put

$$U' := \min(V, U)\ ; \tag{2.39}$$

one of our aims is to show that $U' = U$, but this will be established only in the second step of the proof. We also define $\mathtt{L}$ as in Eq. (2.12), but on the domain $[0, U'/\varepsilon)$.

Step 1. One has

$$|\mathtt{L}(t)| < \mathfrak{n}(\varepsilon t) \qquad \text{for all } t \in [0, U'/\varepsilon) \ . \tag{2.40}$$

To show this, we write the integral inequality (2.37) with $\tau = \varepsilon t$, $\tau' = \varepsilon t'$; this gives

$$\mathfrak{n}(\varepsilon t) > \alpha(\varepsilon t, \varepsilon \mathfrak{n}(\varepsilon t)) + \varepsilon^2 |\mathtt{R}(\varepsilon t)| \int_0^t dt' \, |\mathtt{R}^{-1}(\varepsilon t')| \, \gamma(\varepsilon t', \varepsilon \mathfrak{n}(\varepsilon t'), \mathfrak{n}(\varepsilon t')) \tag{2.41}$$

for all $t \in [0, U/\varepsilon)$, and *a fortiori* for $t \in [0, U'/\varepsilon)$.

On the other hand, Lemma 2.2 can be applied with the constant U therein replaced by U', because $(\mathtt{I}, \Theta)$ is defined on $[0, U'/\varepsilon)$ and $\mathtt{J}$ is defined on $[0, U')$; thus, Eq. (2.31) for $|\mathtt{L}(t)|$ holds for $t \in [0, U'/\varepsilon)$. Now, we apply Lemma 2.3 with

$$T := \frac{U'}{\varepsilon} \ , \qquad \delta(t) := \rho(\varepsilon t)/\varepsilon, \tag{2.42}$$

$$\xi(t, \ell) := \alpha(\varepsilon t, \varepsilon \ell) \ , \quad \eta(t, t', \ell) := \varepsilon^2 \, |\mathtt{R}(\varepsilon t)| \, |\mathtt{R}^{-1}(\varepsilon t')| \, \gamma(\varepsilon t', \varepsilon \ell, \ell) \ ,$$
$$\mathfrak{l}(t) := |\mathtt{L}(t)| \ , \qquad \mathfrak{v}(t) := \mathfrak{n}(\varepsilon t) \ ;$$

of course, the initial condition $\mathfrak{l}(0) = 0$ holds because $\mathtt{I}(0) = I_0 = \mathtt{J}(0)$. Lemma 2.3 gives $\mathfrak{l}(t) < \mathfrak{v}(t)$, which is just the relation (2.40).

Step 2. It is

$$U' = U \tag{2.43}$$

(thus $(\mathtt{I}, \Theta)$ *exists on* $[0, U/\varepsilon)$*, and the inequality of Step 1 holds on this interval).*

It suffices to show that $V \geqslant U$; to this purpose we suppose $V < U$, and infer a contradiction. Indeed, let us put

$$K := \{(t, I) \in [0, V/\varepsilon] \times \mathbf{R}^d \mid |I - \mathtt{J}(\varepsilon t)| \leqslant \varepsilon \mathfrak{n}(\varepsilon t)\} \ . \tag{2.44}$$

This is a closed subset of $\mathbf{R} \times \mathbf{R}^d$; it is bounded, since $t \mapsto \mathtt{J}(\varepsilon t)$, $t \mapsto \mathfrak{n}(\varepsilon t)$ are bounded functions on $[0, V/\varepsilon]$. Thus, K is a compact subset of $\mathbf{R} \times \mathbf{R}^d$. We note that $(t, I) \in K$ implies $I \in \overline{B}(\mathtt{J}(\varepsilon t), \varepsilon \mathfrak{n}(\varepsilon t)) \subset B(\mathtt{J}(\varepsilon t), \rho(\varepsilon t)) \subset \Lambda$ (recall Eq.s (2.36) and (2.20)); thus, $K \subset [0, V/\varepsilon] \times \Lambda$.

The previous considerations ensure compactness of $K \times \mathbf{T} \subset \mathbf{R} \times \Lambda \times \mathbf{T}$; due to Step 1, we have graph $(\mathtt{I}, \Theta) \subset K \times \mathbf{T}$. The inclusion into a compact set and a standard continuation principle for ordinary differential equations [9] imply that the solution $(\mathtt{I}, \Theta)$ can be extended to an interval larger than $[0, V/\varepsilon)$. This contradicts our maximality assumption, and concludes the proof. $\diamond$

2F. A differential reformulation of the previous results. For practical applications, and especially for the numerical implementation of our scheme by standard

packages, it is convenient to replace the integral inequality (2.37) for $\mathfrak{n}$ with a differential equation related to it. This equation is presented hereafter, and will be the basis of all applications discussed in the next sections; it is supplemented by an initial condition, defined implicitly by a fixed point problem.
In the sequel we keep the assumptions at the beginning of paragraph 2E, but we require some more regularity on the functions a, b, c, d, e fulfilling Eq.s (2.23–2.27), namely,

$$a, b \in C^2(\Gamma_\rho, \mathbf{R}) \ , \qquad c, d, e \in C^1(\Gamma_\rho, \mathbf{R}) \ ; \tag{2.45}$$

so, the functions α, γ in Eq.s (2.29) (2.30) are, respectively, of class C^2 and C^1.

2.5 ***Proposition.*** *i) Assume there are real numbers $\ell_*, M \geqslant 0$ and $\sigma > 0$ such that*

$$\Sigma := [\ell_* - \sigma, \ell_* + \sigma] \subset (0, \rho(0)/\varepsilon) \ , \tag{2.46}$$

$$M < 1/\varepsilon \ , \qquad \left|\frac{\partial \alpha}{\partial r}(0, \varepsilon \ell)\right| \leqslant M \quad \text{for } \ell \in \Sigma \ , \tag{2.47}$$

$$|\alpha(0, \varepsilon \ell_*) - \ell_*| + \varepsilon M \sigma < \sigma \ . \tag{2.48}$$

Then, the map $\ell \mapsto \alpha(0, \varepsilon\ell)$ sends the interval Σ into itself and is therein contractive with Lipschitz constant εM. So, there is a unique $\ell_0 \in \Sigma$ solving the fixed point equation

$$\alpha(0, \varepsilon \ell_0) = \ell_0 \ . \tag{2.49}$$

ii) With ℓ_0 as above, let $\mathfrak{m}, \mathfrak{n} \in C^1([0, U), \boldsymbol{R})$ solve the Cauchy problem

$$\frac{d\mathfrak{m}}{d\tau} = |\mathtt{R}^{-1}| \, \gamma(\cdot, \varepsilon \mathfrak{n}, \mathfrak{n}) \ , \qquad \mathfrak{m}(0) = 0 \ , \tag{2.50}$$

$$\frac{d\mathfrak{n}}{d\tau} = \Big(1 - \varepsilon \frac{\partial \alpha}{\partial r}(\cdot, \varepsilon \mathfrak{n})\Big)^{-1} \left(\frac{\partial \alpha}{\partial \tau}(\cdot, \varepsilon \mathfrak{n}) + \varepsilon |\mathtt{R}| |\mathtt{R}^{-1}| \, \gamma(\cdot, \varepsilon \mathfrak{n}, \mathfrak{n}) + \varepsilon |\mathtt{R}|^{-1} \, \Big(\mathtt{R} \bullet \frac{d\mathtt{R}}{d\tau}\Big) \, \mathfrak{m}\right) \ ,$$

$$\mathfrak{n}(0) = \ell_0 \ , \tag{2.51}$$

with the domain conditions

$$0 < \mathfrak{n} < \rho/\varepsilon \ , \qquad \frac{\partial \alpha}{\partial r}(\cdot, \varepsilon \mathfrak{n}) < 1/\varepsilon \tag{2.52}$$

(note that (2.50) implies $\mathfrak{m} \geqslant 0$; in the above, $\bullet$ is the inner product of Eq. (2.2)). Then, the solution $(\mathtt{I}, \Theta)$ of the perturbed system exists on $[0, U/\varepsilon)$ and (with $\mathtt{L}$ as in (2.12))

$$|\mathtt{L}(t)| \leqslant \mathfrak{n}(\varepsilon t) \qquad \text{for all } t \in [0, U/\varepsilon). \tag{2.53}$$

Proof. It is found in the Appendix C, after a necessary lemma. $\diamond$

3 A summary of the method, and how to test it.

3A. The main steps to implement the scheme of the previous section. In the approach we have outlined, the steps to be performed are the following ones.
i) Compute $\overline{f}$ and the functions $s, p, ..., \mathscr{M}, \mathscr{G}, \mathscr{H}$ of Eq.s (2.8) (2.14–2.19).
ii) Determine the solution $\mathtt{J}$ of Eq. (1.6), on some interval $[0, U)$; solve Eq.s (2.10) (2.11) for $\mathtt{R}, \mathtt{K}$ on the same interval.
iii) Find a set of functions ρ, a, b, c, d, e as in paragraph 2C, so as to fulfil the inequalities (2.23–2.27); from them, define the functions α, γ via Eq.s (2.29) (2.30). In the subsequent steps, we make on $a, ..., e$ the assumptions (2.45).
iv) Determine ℓ_0, solving the fixed point problem (2.49).
v) Search for functions $\mathfrak{m}, \mathfrak{n}$ fulfillfing Eq.s (2.50) (2.51), with the domain conditions (2.52). If these equations and (1.6) have solutions on some interval $[0, U)$, we can grant existence on $[0, U/\varepsilon)$ for the solution $(\mathtt{I}, \Theta)$ of (1.3), and we know that $\mathtt{L}(t) := (\mathtt{I}(t) - \mathtt{J}(\varepsilon t))/\varepsilon$ fulfils on this interval the bound $|\mathtt{L}(t)| \leqslant \mathfrak{n}(\varepsilon t)$.

Here are some general comments on the practical implementation of the previous steps (these will also be useful to introduce the examples of the next section).
i) Of course, the computation of $\overline{f}, s, p, ..., \mathscr{M}$ is more or less difficult depending on f, g and ω, concerning especially the integrals over ϑ. These computations can involve special functions (it should be noted that, in many examples coming from mechanics, f, g and ω are themselves special functions). Generally, the determination of $\overline{f}, s, p, ..., \mathscr{M}$ is simple when, for fixed I, f and g are trigonometric polynomials in ϑ. Concerning $\mathscr{G}$ and $\mathscr{H}$, see the remarks that follow Eq.s (2.5) (2.6).
ii) The determination of $\mathtt{J}, \mathtt{R}, \mathtt{K}$ will be analytical in the symplest cases, and otherwise numerical.
iii) For the implementation of our scheme, the functions b, c, d, e are slightly less important than a; in fact, they are always multiplied by the small parameter ε whenever they appear in steps iii) iv) v). For this reason, it is important to compute a estimating as accurately as possible the left hand side in Eq. (2.23); as for $b, ..., e$, in many cases one can accept rougher majorizations for the left hand sides of Eq.s (2.24–2.27).
In many applications, such as in the examples of the next section, the functions $a, b, .., e$ will have the form

$$a(\tau, r) := \widehat{a}(\mathtt{J}(\tau), \mathtt{R}(\tau), \mathtt{K}(\tau), r), \;\; b(\tau, r) := \widehat{b}(\mathtt{J}(\tau), r), ..., \;\; e(\tau, r) = \widehat{e}(\mathtt{J}(\tau), r) \tag{3.1}$$

depending on certain known functions

$$\widehat{a} \in C^2(\widehat{\Delta}, \mathbf{R}) \; , \quad \widehat{b} \in C^2(\widehat{\Upsilon}, \mathbf{R}) \; , \quad \widehat{c}, \widehat{d}, \widehat{e} \in C^1(\widehat{\Upsilon}, \mathbf{R}) \; , \tag{3.2}$$

with domains

$$\widehat{\Delta} \subset \mathbf{R}^d \times \mathrm{T}^1_1(\mathbf{R}^d) \times \mathbf{R}^d \times \mathbf{R} \text{ open}, \quad \widehat{\Upsilon} \subset \mathbf{R}^d \times \mathbf{R} \text{ open} \quad \text{such that} \tag{3.3}$$

$$(\mathtt{J}(\tau), \mathtt{R}(\tau), \mathtt{K}(\tau), r) \in \widehat{\Delta} \ , \qquad (\mathtt{J}(\tau), r) \in \widehat{\Upsilon} \quad \text{for all} \quad (\tau, r) \in \Gamma_\rho \ .$$

Of course, in this case it is

$$\alpha(\tau, r) = \widehat{\alpha}(\mathtt{J}(\tau), \mathtt{R}(\tau), \mathtt{K}(\tau), r) \ , \quad \gamma(\tau, r, \ell) = \widehat{\gamma}(\mathtt{J}(\tau), r, \ell) \ , \tag{3.4}$$

where $\widehat{\alpha} \in C^2(\widehat{\Delta}, \mathbf{R})$ and $\widehat{\gamma} \in C^1(\widehat{\Upsilon} \times \mathbf{R}, \mathbf{R})$ are defined by

$$\widehat{\alpha}(J, \mathscr{R}, K, r) := \widehat{a}(J, \mathscr{R}, K, r) + \varepsilon \widehat{b}(J, r) \ , \tag{3.5}$$

$$\widehat{\gamma}(J, r, \ell) := \widehat{c}(J, r) + \widehat{d}(J, r)\ell + \frac{1}{2}\widehat{e}(J, r)\ell^2 \ . \tag{3.6}$$

Furthermore, the derivative $\partial\alpha/\partial\tau$ in Eq. (2.51) is given by

$$\frac{\partial \alpha}{\partial \tau}(\cdot, r) = \frac{\partial \widehat{\alpha}}{\partial J}(\mathtt{J}, \mathtt{R}, \mathtt{K}, r) \bullet \frac{d\mathtt{J}}{d\tau} + \frac{\partial \widehat{\alpha}}{\partial \mathscr{R}}(\mathtt{J}, \mathtt{R}, \mathtt{K}, r) \bullet \frac{d\mathtt{R}}{d\tau} + \frac{\partial \widehat{\alpha}}{\partial K}(\mathtt{J}, \mathtt{R}, \mathtt{K}, r) \bullet \frac{d\mathtt{K}}{d\tau} \tag{3.7}$$

with $\partial\widehat{\alpha}/\partial\mathscr{R} := (\partial\widehat{\alpha}/\partial\mathscr{R}^i_j)$, etc. In these situations, the function $\tau \mapsto \rho(\tau)$ determining the domain of $a, ..., e$ will often depend on τ through $\mathtt{J}$, i.e., $\rho(\tau) = \widehat{\rho}(\mathtt{J}(\tau))$. The structure (3.1) for a, b, etc. appears naturally in cases where these functions can be obtained maximizing the left hand sides of Eq.s (2.23), (2.24), etc. by analytical means.

In more complicated situations, one could consider the possibility to determine a, b, etc., maximising the left hand sides of Eq.s (2.23), (2.24), etc. by numerical (or partially numerical) techniques. These would give tables of numerical maxima, to be subsequently interpolated by elementary functions to get a, b, etc. . A second possibility is to derive the evolution equation for the maximum points of interest as function of τ, to be coupled with the other differential equations in our general framework; this approach should work if there are no bifurcations.

Both possibilities outlined above are especially interesting for the function a, since this requires the greatest accuracy; however, they will be investigated elsewhere.

iv) The fixed point ℓ_0 in (2.49) is given by the standard iterative formula $\ell_0 = \lim_{n\to+\infty} l_n$, where $l_n := \alpha(0, \varepsilon l_{n-1})$ and l_1 is chosen arbitrarily in Σ. One can compute numerically the sequence (l_n) up to a sufficiently large value $n = N$, and then assume $\ell_0 \simeq l_N$. ([2])

v) Even in cases where all the other functions have known analytical expressions, the differential equations (2.50) (2.51) for $\mathfrak{m}, \mathfrak{n}$ will be typically too difficult to be solved analytically. So, a numerical treatment will be necessary.

If we do not have analytical expressions for $\mathtt{J}, \mathtt{R}, \mathtt{K}$, it may be convenient to regard Eq.s (1.6) (2.10) (2.11) (2.50) (2.51) as a coupled system for the unknowns $\mathtt{J}, \mathtt{R}, \mathtt{K}, \mathfrak{m}, \mathfrak{n}$, to be solved numerically on a chosen interval $[0, U)$.

[2] By the standard theory of contractions, $|\ell_0 - l_N| \leqslant (\varepsilon M)^{N-1}|l_2 - l_1|/(1 - \varepsilon M)$, where M is the constant in Proposition 2.5.

3B. The "$\mathfrak{N}$-operation". Let us fix the attention on the simple situations where the functions $\overline{f}, s, ..., \mathscr{H}$ have known analytical expressions and a, b, c, d, e have the form (3.1), depending on known functions $\widehat{a}, ..., \widehat{e}$. It is not difficult to write a program of general use for these situations, which computes the fixed point ℓ_0 and the functions $\mathtt{J}$, $\mathtt{R}$, $\mathtt{K}$, $\mathfrak{m}$, $\mathfrak{n}$ solving numerically the equations (2.49) (1.6) (2.11) (2.50) (2.51). From now on, the computation of $\ell_0, \mathtt{J}, ..., \mathfrak{n}$ by such a program, for given $\overline{f}, ..., \widehat{e}$ (and I_0, ϑ_0, U), will be referred to as the $\mathfrak{N}$-*operation*. Of course, the main outcomes of this operation are the solution $\mathtt{J}$ of the averaged system and the function $\mathfrak{n}$ binding $|\mathtt{L}(t)|$.

We have written a general program for the above purpose, using the MATHEMATICA system. Concerning Eq.(2.51) for $\mathfrak{n}$, in this program the derivative $\partial\alpha/\partial\tau$ is expressed via Eq. (3.7); the derivatives $d\mathtt{J}/d\tau, d\mathtt{K}/d\tau$ and $d\mathtt{R}/d\tau$ which occur in (3.7) and (2.51) are expressed via Eq.s (1.6) (2.10) (2.11) (MATHEMATICA is also useful, in the symbolic mode, to produce the input of the above program, i.e., the functions $\overline{f}, ..., \widehat{e}$; this will appear from the examples of the next section).

3C. Testing the effectiveness of the previous method: the "$\mathfrak{L}$-operation". By the $\mathfrak{L}$-operation we mean, essentially, the computation of $\mathtt{L}$ by direct numerical solution of the perturbed system on $[0, U/\varepsilon)$. To avoid misunderstandings, we stress that in the present framework the purpose of the $\mathfrak{L}$-operation is merely to check the reliability of the estimate $|\mathtt{L}(t)| \leqslant \mathfrak{n}(\varepsilon t)$ produced by the $\mathfrak{N}$-operation, and to prove quantitatively that the direct solution of the perturbed system is generally much slower than $\mathfrak{N}$. When U/ε is very large, the $\mathfrak{L}$-operation may be impossible within reasonable times; an example will be given in the next Section (see Figure 3f, and the explanations for it). Of course, the main usefulness of the $\mathfrak{N}$-operation is just the treatment of these cases!

To be precise, the $\mathfrak{L}$-operation is the numerical determination of $\mathtt{J}, \mathtt{L}, \Theta$ in the following way. First, the function $\tau \in [0, U) \to \mathtt{J}(\tau)$ is obtained solving the averaged system (1.6) for $\mathtt{J}$; then, the functions $t \in [0, U/\varepsilon) \to \mathtt{L}(t), \Theta(t)$ are determined solving their exact evolution equations derived from (1.3) (1.6), i.e.,

$$\begin{cases} (d\mathtt{L}/dt)(t) = f(\mathtt{J}(\varepsilon t) + \varepsilon\mathtt{L}(t), \Theta(t)) - \overline{f}(\mathtt{J}(\varepsilon t)), & \mathtt{L}(0) = 0 \ , \\ (d\Theta/dt)(t) = \omega(\mathtt{J}(\varepsilon t)) + \varepsilon g(\mathtt{J}(\varepsilon t) + \varepsilon\mathtt{L}(t), \Theta(t)), & \Theta(0) = \vartheta_0 \ . \end{cases} \tag{3.8}$$

It is easy to write a MATHEMATICA program that computes numerically $\mathtt{J}, \mathtt{L}, \Theta$ for given f, g, ω, I_0, ϑ_0 .

When the $\mathfrak{L}$-operation can performed within reasonable times, it can be used to test the $\mathfrak{N}$-procedure along these lines:

i) one compares the graph of the estimator $\mathfrak{n}$ (an $\mathfrak{N}$-output) with the graph of the function $|L|$ (an $\mathfrak{L}$- output);

ii) one also compares the CPU times $\mathfrak{T}_{\mathfrak{N}}$, $\mathfrak{T}_{\mathfrak{L}}$ for the two operations.

These tests are presented in the next section; they are based on the programs mentioned here and in paragraph 3B. In most examples, the estimator $\mathfrak{n}$ practically

coincides with the envelope of the rapidly oscillating graph of $|\mathtt{L}|$; furthermore, $\mathfrak{T}_{\mathfrak{N}}$ is generally smaller than $\mathfrak{T}_{\mathfrak{L}}$ by one or more orders of magnitude.

4 Examples.

In any example we consider, the initial condition for the angle is always

$$\vartheta_0 := 0 \ . \tag{4.1}$$

Given f, g and ω, the functions $\overline{f}, s, ..., \mathscr{G}, \mathscr{H}$ and $\rho, a, ..., e$ are computed explicitly for all I_0 (and U). After this, specific choices are made for I_0, U and ε, and the $\mathfrak{N}$-operation is performed; to test the accuracy of the method, the $\mathfrak{L}$-operation is also performed and some comparisons are made, as suggested at the end of the previous section. The results are summarized in the figures which conclude the section. Each figure gives the graph of the estimator $\mathfrak{n}(\tau)$ provided by $\mathfrak{N}$ for $\tau \in [0, U)$; it also gives the graph of $|\mathtt{L}(\tau/\varepsilon)|$ in the same interval (except one case, where $\mathfrak{L}$ has not been possible within reasonable times).
Figures referring to an example are labelled by the same number and by a letter (so, Fig.s 1a, 1b and 1c refer to Example 1). The legend of each figure specifies the choices of I_0, ε, U, and the CPU times $\mathfrak{T}_{\mathfrak{N}}$, $\mathfrak{T}_{\mathfrak{L}}$ (in seconds) in the execution of the two operations ([3]).
In the chosen examples, one derives simple analytical expressions for the functions $\mathtt{J}, \mathtt{R}, \mathtt{K}$ but not for $\mathfrak{m}, \mathfrak{n}$. However, with the view of a general comparison between the $\mathfrak{N}$- and $\mathfrak{L}$-operations, all examples have been treated by the general MATHEMATICA programs mentioned in paragraphs 3B-3C, which solve numerically all the differential equations involved. Therefore, the reported times $\mathfrak{T}_{\mathfrak{N}}$, $\mathfrak{T}_{\mathfrak{L}}$ include contributions from the determination of $\mathtt{J}, \mathtt{R}, \mathtt{K}$. In any case, the analytical expressions of these functions are written for completeness.
For each example:
i) the auxiliary functions $s, ..., \mathscr{H}, \rho, a, ..., e$ are reported in a table. All the related computations are analytical; the most lengthy have been performed using MATHEMATICA in the symbolic mode.
ii) The function ρ always gives the distance of $\mathtt{J}(\tau)$ from the boundary of the actions space Λ.
iii) Some details on the computation of the functions a and b, c are given in the Appendices D and E, respectively. The expressions for d, e follow trivially from the ones for $\mathscr{G}, \mathscr{H}$ in the corresponding tables.

Example 1: the van der Pol equation. This is a system of the form (1.3) for $(\mathtt{I}, \Theta)$, with

$$d := 1 \ , \qquad \Lambda := (0, +\infty) \ , \qquad \omega(I) := -1 \ , \tag{4.2}$$

[3] of course these times, depending on the PC employed, are merely indicative.

$$f(I,\vartheta) := I(1-\frac{I}{2}) - I\cos(2\vartheta) + \frac{I^2}{2}\cos(4\vartheta), \quad g(I,\vartheta) := \frac{1-I}{2}\sin(2\vartheta) - \frac{I}{4}\sin(4\vartheta)\ .$$

The functions $\mathtt{x} := \sqrt{2\mathtt{I}}\cos\Theta$, $\mathtt{v} := \sqrt{2\mathtt{I}}\sin\Theta$ fulfil the equations $\dot{\mathtt{x}} = \mathtt{v}$, $\dot{\mathtt{v}} = -\mathtt{x} - \varepsilon\,(\mathtt{x}^2-1)\mathtt{v}$, yielding the familiar van der Pol equation $\ddot{\mathtt{x}} + \mathtt{x} + \varepsilon\,(\mathtt{x}^2-1)\dot{\mathtt{x}} = 0$. It is found that

$$\overline{f}(I) = I(1-\frac{I}{2})\ ; \tag{4.3}$$

the auxiliary functions $s, v, ..., \mathscr{H}$ of paragraph 2B are reported in Table 1 ([4]). The averaged system (1.6) has the solution

$$\mathtt{J}(\tau) = \frac{2I_0}{I_0 + (2-I_0)\,e^{-\tau}} \tag{4.4}$$

for $\tau \in [0,+\infty)$, tending to 2 for $\tau \to +\infty$: this long time behavior is the manifestation, in the averaging approximation, of the well known limit cycle of the van der Pol equation ($\mathtt{J}(\tau)$ also exists for some or all $\tau < 0$, but we are not interested in this fact). The Cauchy problems (2.10), (2.11) for the unknown real functions $\mathtt{R}, \mathtt{K}$ have solutions

$$\mathtt{R}(\tau) = \frac{4e^{-\tau}}{(I_0 + (2-I_0)\,e^{-\tau})^2}\ , \qquad \mathtt{K}(\tau) = 0 \tag{4.5}$$

for $\tau \in [0,+\infty)$. From now on, τ is confined to an interval $[0,U)$ (and, of course, U will be chosen finite in the subsequent numerical computations).

Our next step is to construct functions $\rho, a, ..., e$ as in paragraph 2C; these are also reported in Table 1 ([5]). All the functions $a, ..., e$ are C^∞ in (τ, r), and non decreasing in r; they have the form $a(\tau,r) = \widehat{a}(\mathtt{J}(\tau),r)$, $b(\tau,r) = \widehat{b}(\mathtt{J}(\tau),r)$, ..., $e(\tau,r) = \widehat{e}(\mathtt{J}(\tau),r)$, where $\widehat{a}, \widehat{b}, \widehat{c}$ are read from the Table and $\widehat{d} := 0$, $\widehat{e} := 1$ everywhere; this corresponds to a special case of Eq. (3.1). Similar remarks could be made for the other Examples, but will be no longer repeated.

Comments on this example and the figures. Figures 1a, 1b, 1c refer to the initial data $I_0 = 1/2$ or $I_0 = 4$, one below and the other above the critical value $I = 2$ (i.e., the limit cycle in the averaging approximation); U is 10 or 200. The ratio $\mathfrak{T}_{\mathfrak{N}}/\mathfrak{T}_{\mathfrak{L}}$ is between 1/150 and 1/40, in the three cases. Due to the limit cycle, one expects

[4] We note that the domain $\Lambda_\dagger$ of $\mathscr{G}, \mathscr{H}$ is made of pairs $(I, \delta I)$ as indicated in Table 1. In all the other examples, $\Lambda_\dagger$ can be read as well from the tables.

[5] The functions b, c constructed in this way could be replaced by appropriate, simpler majorants reducing the "confidence interval" $[0, \mathtt{J}(\tau))$ for r; for example, one could redefine $\rho(\tau) := \min(\mathtt{J}(\tau), 1/10)$ and infer upper bounds for b, c by means of the inequalities $r^k \leqslant r/10^{k-1}$, for $k = 2, 3, ...$, holding for $r \in [0, \rho(\tau))$. These upper bounds are fairly simple, since they depend linearly on r; of course, their use is correct if one checks a posteriori that $0 < \varepsilon\,\mathfrak{n}(\tau) < \min(\mathtt{J}(\tau), 1/10)$ for all $\tau \in [0,U)$. However, to perform the $\mathfrak{N}$-operation in all cases presented in the figures we have used directly the complicated expressions in Table I, since these are easily handled by MATHEMATICA.

Table 1. Auxiliary functions for Example 1.

For $I \in (0, +\infty)$, $\vartheta \in \mathbf{T}$ and $\delta I \in (-I, +\infty)$:

$$s(I,\vartheta) = \frac{I}{8}\Big(4\sin(2\vartheta) - I\sin(4\vartheta)\Big)\ , \qquad v(I,\vartheta) = -\frac{I}{32}\Big(8 - I - 8\cos(2\vartheta) + I\cos(4\vartheta)\Big)\ ,$$

$$p(I,\vartheta) = \frac{I}{8}\Big((4 - 2I - I^2)\sin(2\vartheta) + I(I-4)\sin(4\vartheta) + I^2\sin(6\vartheta)\Big), \qquad \overline{p}(I) = 0\ ,$$

$$q(I,\vartheta) = -\frac{I}{32}\Big(16 - 10I + 2I^2 - (16 - I^2)\cos(2\vartheta) + I(10 - 2I)\cos(4\,\vartheta) - I^2\cos(6\vartheta)\Big)\ ,$$

$$w(I,\vartheta) = -\frac{I}{96}\Big(24 - 24I - I^2 - 6(4 - 2I - I^2)\cos(2\vartheta) + 3I(4 - I)\cos(4\vartheta) - 2I^2\cos(6\vartheta)\Big)\ ,$$

$$u(I,\vartheta) = -\frac{I}{128}\Big(64 - 120I + 36I^2 + I^3 + (-64 + 64I + 50I^2 - 12I^3)\cos(2\vartheta) + 4I(14 - 17I - I^2)\cos(4\vartheta) + 6I^2(-3 + 2I)\cos(6\vartheta) + 3I^3\cos(8\vartheta)\Big)\ ,$$

$$\mathscr{M}(I) = -1 + I - \frac{1}{2}I^2\ , \qquad \mathscr{G}(I,\delta I) = 0, \qquad \mathscr{H}(I,\delta I) = -1\ .$$

For $\tau \in [0, U)$, $\rho(\tau) := \mathtt{J}(\tau)$.

For $\tau \in [0, U)$ and $r \in [0, J(\tau))$:

$$a(\tau, r) := \frac{1}{8}\Big(-2 + 10(J+r)^2 + (J+r)^4 + 2(1 + 2(J+r)^2)^{3/2}\Big)^{1/2}_{J=\mathtt{J}(\tau)}\ ,$$

$$b(\tau, r) := \frac{1}{96}\Big(120J^6 + 12J^5(23 + 56r) + 3J^4(192 + 474r + 517r^2) + 12J^3r(72 + 180r + 157r^2) + 6J^2r^2(372 + 530r + 231r^2) + 12Jr^3(216 + 213r + 46r^2) + r^4(1404 + 690r + 91r^2)\Big)^{1/2}_{J=\mathtt{J}(\tau)}\ ,$$

$$c(\tau, r) := \frac{1}{384}\Big(6512J^8 + 24J^7(671 + 2096r) + 24J^6(1693 + 5484r + 6956r^2) + 8J^5 \times (1812 + 31188r + 39375r^2 + 38726r^3) + 12J^4(768 + 4436r + 61358r^2 + 37966r^3 + 29997r^4) + 8J^3r(4680 + 39948r + 125584r^2 + 62193r^3 + 35046r^4) + 12J^2r^2(1824 + 52152r + 61180r^2 + 37311r^3 + 12021r^4) + Jr^3(119808 + 445536r + 425592r^2 + 210995r^3 + 41976r^4) + 4r^4(21600 + 33024r + 30127r^2 + 10383r^3 + 1377r^4)\Big)^{1/2}_{J=\mathtt{J}(\tau)},$$

$$d(\tau, r) := 0\ , \qquad e(\tau, r) := 1\ .$$

$|L(\tau/\varepsilon)|$ to be bounded on the whole interval $[0, +\infty)$; this fact is reproduced very well by our estimator $\mathfrak{n}(\tau)$, that appears to approach a constant value for large τ (see in particular Figure 1c).

Example 2: a case with action-dependent frequency. We choose

$$d = 1 \ , \qquad \Lambda := (0, +\infty) \ , \qquad \omega(I) := I \ , \tag{4.6}$$

$$f(I, \vartheta) := \kappa I^2(1 - \cos(2\vartheta)) \ , \qquad g(I, \vartheta) := \kappa I^2(1 + \cos(2\vartheta)) \ , \qquad \kappa \in \{\pm 1\} \ .$$

It is

$$\overline{f}(I) = \kappa I^2 \ , \tag{4.7}$$

and the auxiliary functions $s, v, ..., \mathscr{H}$ are reported in Table 2. Let us comment on the vanishing of ω for $I \to 0$. Our framework shows this "resonance" to be false: in fact, even though Eq.s (2.8) (2.14) (2.16) for s, v, w contain a factor $1/\omega$, in this case none of these functions is singular for $I \to 0$, since f, g vanish in this limit more rapidly than ω.

The averaged system (1.6) is fulfilled with

$$\mathtt{J}(\tau) = \frac{I_0}{1 - \kappa\tau I_0} \ \text{ for } \tau \in [0, W_{\kappa, I_0}), \quad W_{\kappa, I_0} := \begin{cases} 1/I_0 & \text{if } \kappa = +1, \\ +\infty & \text{if } \kappa = -1. \end{cases} \tag{4.8}$$

Eq.s (2.10) (2.11) for $\mathtt{R}, \mathtt{K}$ have solutions

$$\mathtt{R}(\tau) = \frac{1}{(1 - \kappa I_0 \tau)^2} \ , \qquad \mathtt{K}(\tau) = \frac{\kappa I_0^2 \log(1 - \kappa I_0 \tau)}{2(1 - \kappa I_0 \tau)^2} \leqslant 0 \tag{4.9}$$

on the same domain. In the sequel we assume $\tau \in [0, U)$, with $U \leqslant W_{\kappa, I_0}$; the functions ρ (the same of Example 1) and a, b, c, d, e are also reported in Table 2.

Comments on this example and the figures. Fig.s 2a, 2b and 2c refer to the case $\kappa = 1$, while Fig.s 2d and 2e refer to $\kappa = -1$; the initial datum is always $I_0 = 1$. The two cases are radically different: in fact, according to Eq. (4.8), the solution $\mathtt{J}(\tau)$ of the averaged system diverges for $\tau \to 1^-$ if $\kappa = 1$, whereas for $\kappa = -1$ it is defined for arbitrarily large τ and vanishes for $\tau \to +\infty$. The figures seem to indicate a similar behaviour for the function $\tau \mapsto |L(\tau/\varepsilon)|$; this behaviour is reproduced very well by our estimator $\mathfrak{n}(\tau)$, which remains close to the envelope of $|L(\tau/\varepsilon)|$ even for $\kappa = 1$ and τ close to 1 (see, in particular, Fig.s 2a and 2c).

Table 2. Auxiliary functions for Example 2.

For $I \in (0, +\infty)$, $\vartheta \in \mathbf{T}$ and $\delta I \in (-I, +\infty)$:

$$s(I,\vartheta) = -\frac{\kappa}{2} I \sin(2\vartheta) \ , \qquad v(I,\vartheta) = -\frac{\kappa}{4}(1-\cos(2\vartheta)) \ ,$$

$$p(I,\vartheta) = -\frac{1}{4}\, I^2 \Big(2I + 4\, I\, \cos(2\vartheta) + 2\, \sin(2\vartheta) + 2I\, \cos(4\,\vartheta) - \sin(4\,\vartheta)\Big), \ \ \overline{p}(I) = -\frac{1}{2} I^3 \ ,$$

$$q(I,\vartheta) = -\frac{1}{4}\, I^2 \Big(2\, \sin(2\vartheta) + \sin(4\,\vartheta)\Big),$$

$$w(I,\vartheta) = -\frac{1}{16} I \Big(3 - 4\, \cos(2\vartheta) + 8I\, \sin(2\vartheta) \ + \cos(4\,\vartheta) + 2I\, \sin(4\,\vartheta)\Big),$$

$$u(I,\vartheta) = -\frac{\kappa}{32}\, I^2 \Big(16\, I^2 + 10 + (40\, I^2 - 15)\, \cos(2\vartheta) + 40 I \sin(2\vartheta) +$$

$$+ (32\, I^2 + 6)\, \cos(4\,\vartheta) - 8\, I\, \sin(4\,\vartheta) + (8\, I^2 - 1) \cos(6\,\vartheta) - 8\, I\, \sin(6\,\vartheta)\Big) \ ,$$

$$\mathscr{M}(I) = 6 I^2 \ , \qquad \mathscr{G}(I, \delta I) := -\frac{1}{2}(3I^2 + 3I\delta I + \delta I^2) \ , \qquad \mathscr{H}(I, \delta I) := 2\kappa \ .$$

For $\tau \in [0, U)$, $\rho(\tau) := \mathtt{J}(\tau)$.

For $\tau \in [0, U)$ and $r \in [0, J(\tau))$:

$$a(\tau, r) := \frac{1}{2}(\mathtt{J}(\tau) + r) - \mathtt{K}(\tau) \ ,$$

$$b(\tau, r) := \frac{1}{8\sqrt{2}} \Big(50 J^4 + (55 + 200r) J^3 + (38 + 85r + 300r^2) J^2 + (65 + 33r + 200r^2) J r +$$

$$+ (32 + 27r + 50r^2) r^2\Big)^{1/2}_{J = \mathtt{J}(\tau)} \ ,$$

$$c(\tau, r) := \frac{1}{16\sqrt{2}} \Big(4608 J^8 + (3904 + 36864r) J^7 + (1520 + 23296r + 129024r^2) J^6 + (1856 +$$

$$+ 5696r + 57792r^2 + 258048r^3) J^5 + (4853 + 5352r + 10032r^2 + 76160r^3 + 322560r^4) J^4 +$$

$$+ (3086 + 7824r + 11008r^2 + 56000r^3 + 258048r^4) J^3 r + (1862 + 2976r + 9808r^2 +$$

$$+ 21504r^3 + 129024r^4) J^2 r^2 + (1024 + 2312r + 5440r^2 + 7168r^3 + 36864r^4) J r^3 +$$

$$+ (512 + 752r + 1296r^2 + 1280r^3 + 4608r^4) r^4\Big)^{1/2}_{J = \mathtt{J}(\tau)} \ ,$$

$$d(\tau, r) := \frac{1}{2}\, (3J^2 + 3Jr + r^2)_{J = \mathtt{J}(\tau)}, \qquad e(\tau, r) := 2 \ .$$

Table 3. Auxiliary functions for Example 3.

For $I \in (0,+\infty)$, $\vartheta \in \mathbf{T}$ and $\delta I \in (-I,+\infty)$:

$$s(I,\vartheta) = -\frac{1}{I}\sin\vartheta \ , \qquad v(I,\vartheta) = -\frac{1}{I^2}(1-\cos\vartheta) \ ,$$
$$p(I,\vartheta) = \frac{1}{2I^2}(2\sin\vartheta - \sin(2\vartheta)) \ , \qquad \overline{p}(I) = 0 \ ,$$
$$q(I,\vartheta) = \frac{1}{I^3}(3 - 4\cos\vartheta + \cos(2\vartheta)) \ , \qquad w(I,\vartheta) = \frac{q(I,\vartheta)}{4} \ ,$$
$$u(I,\vartheta) = \frac{3}{8I^4}\,(-10 + 15\cos\vartheta - 6\cos(2\vartheta) + \cos(3\vartheta)) \ ,$$
$$\mathscr{M}(I) = 0 \ , \qquad \mathscr{G}(I,\delta I) := 0 \ , \qquad \mathscr{H}(I,\delta I) := 0 \ .$$

For $\tau \in [0,U)$, $\rho(\tau) := \mathtt{J}(\tau)$.

For $\tau \in [0,U)$ and $r \in [0,J(\tau))$:

$$a(\tau,r) := \frac{1}{\mathtt{J}(\tau) - r} \ , \qquad b(\tau,r) := \frac{2}{(\mathtt{J}(\tau)-r)^3} \ , \qquad c(\tau,r) := \frac{12}{(\mathtt{J}(\tau)-r)^4} \ ,$$
$$d(\tau,r) := 0 \ , \qquad e(\tau,r) := 0 \ .$$

Example 3: a truly resonant case. Let us pass to a case where the vanishing of ω for $I \to 0$ gives rise to singularities for s, v, w and other auxiliary functions. We assume

$$d = 1 \ , \qquad \Lambda := (0,+\infty) \ , \qquad \omega(I) := I \ , \tag{4.10}$$
$$f(I,\vartheta) := 1 - \cos\vartheta \ , \qquad g(I,\vartheta) := 0 \ .$$

This example is considered in [4] [7] to introduce the subject of resonances; it is inspired by a two-frequency example in [2]. In this case,

$$\overline{f}(I) = 1 \ ; \tag{4.11}$$

the functions $s, ..., \mathscr{G}, \mathscr{H}$ are reported in Table 3. The averaged system (1.6) has the solution

$$\mathtt{J}(\tau) = I_0 + \tau \tag{4.12}$$

for $\tau \in [0,+\infty)$. Eq.s (2.10) (2.11) for $\mathtt{R}$, $\mathtt{K}$ are very simple in this case, since $\dfrac{\partial \overline{f}}{\partial I} = 0$ and $\overline{p} = 0$; this implies

$$\mathtt{R}(\tau) = 1 \ , \qquad \mathtt{K}(\tau) = 0 \ . \tag{4.13}$$

From now on, $\tau \in [0,U)$; the functions ρ, a, b, c, d, e are reported in Table 3.

Comments on this example and the figures. The resonance for $I \to 0^+$ could be expected to give problems for initial data I_0 close to zero (these problems should appear mainly for small τ, since Eq. (4.12) for $\mathtt{J}$ shows a departure from the resonance as τ grows). As a matter of fact, the estimator $\mathfrak{n}$ approximates well the

envelope of $|\mathtt{L}(\tau/\varepsilon)|$ even for small τ and data fairly close to zero, such as $I_0 = 1/2$: the agreement is rather good for $\varepsilon = 10^{-2}$ (Fig.s 3a and 3b) and very good for $\varepsilon = 10^{-3}$ (Fig.s 3c and 3d).
The agreement between $\mathfrak{n}$ and the envelope of $|\mathtt{L}|$ is very good even for $\varepsilon = 10^{-2}$, if we consider the larger datum $I_0 = 2$ (Fig.3e). Fig.3f refers to the same situation on the larger interval $\tau \in [0, 200)$. The statement on $\mathfrak{T}_{\mathfrak{L}}$ in the legend means that the numerical computation of $\mathtt{L}$ was interrupted after 240 seconds, when the package had not yet produced a result; note that, on the contrary, the $\mathfrak{N}$-operation for the same interval is very fast.

Example 4: damped Euler's top. We consider the system (1.3), with

$$d = 2, \qquad \Lambda := \{I = (I^1, I^2) \mid I^1, I^2 \in (0, +\infty)\}, \qquad \omega(I) = I^1 I^2, \tag{4.14}$$

$$f(I, \vartheta) := \big(-I^1(\lambda_1 + \mu\cos(2\vartheta)), -I^2(\lambda_2 - \mu\cos(2\vartheta)\big) \ , \quad g(I, \vartheta) := \mu\sin(2\vartheta) \ ;$$

this depends on three real coefficients $\mu, \lambda_1, \lambda_2$ for which we assume

$$\lambda_1 > 0 \ , \qquad -\lambda_1 < \mu < \lambda_1 \ , \qquad \lambda_2 > -\lambda_1 \ . \tag{4.15}$$

This system is related to Euler's equations for the components $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ of the angular velocity of an axially symmetric top, in presence of weak damping. More precisely, assume that the moment of the damping forces is a linear function of the angular velocity, and that the linear operator expressing this dependence has a diagonal matrix $-\varepsilon\,\mathrm{diag}(E, F, G)$ in the reference system in which the inertia operator has the form $\mathrm{diag}(A, A, C)$, with $A, C, E, F, G, \varepsilon \in (0, +\infty)$ ([6]). Then, Euler's equations are

$$A\dot{\mathfrak{p}} + (C - A)\mathfrak{q}\mathfrak{r} = -\varepsilon E\mathfrak{p} \ , \quad A\dot{\mathfrak{q}} - (C - A)\mathfrak{p}\mathfrak{r} = -\varepsilon F\mathfrak{q} \ , \quad C\dot{\mathfrak{r}} = -\varepsilon G\mathfrak{r} \ . \tag{4.16}$$

Given this system, we define $\mu, \lambda_1, \lambda_2$ through the equations

$$E = A(\mu + \lambda_1) \ , \qquad F = A(\lambda_1 - \mu) \ , \qquad G = C(\lambda_1 + \lambda_2) \ , \tag{4.17}$$

which imply the inequalities (4.15). Now, if $(\mathtt{I}, \Theta) = (\mathtt{I}^1, \mathtt{I}^2, \Theta)$ is such that $\dot{\mathtt{I}} = \varepsilon f(\mathtt{I}, \Theta)$ and $\dot{\Theta} = \omega(\mathtt{I}) + \varepsilon g(\mathtt{I}, \Theta)$, the functions

$$\mathfrak{p} := \mathtt{I}^1\cos\Theta \ , \qquad \mathfrak{q} := \mathtt{I}^1\sin\Theta \ , \qquad \mathfrak{r} := \frac{A}{C - A}\mathtt{I}^1\mathtt{I}^2 \tag{4.18}$$

fulfil Euler's equations (4.16).
Let us return to (4.14). This implies

$$\overline{f}(I) = (-\lambda_1 I^1, -\lambda_2 I^2) \ ; \tag{4.19}$$

[6] Of course, quantities like $A, ..., G$, the time t, etc., can be treated as real numbers, because we suppose to have fixed all the necessary physical units.

the functions $s, ..., \mathscr{G}, \mathscr{H}$ are reported in Table 4. The averaged system has the solution

$$\mathtt{J}^i(\tau) = I_0^i \, e^{-\lambda_i \tau} \qquad (i = 1, 2) \tag{4.20}$$

for $\tau \in [0, +\infty)$. Eq.s (2.10) (2.11) for the 2×2 matrix function $\mathtt{R}$ and for the 2-component function $\mathtt{K}$ have the solutions

$$\mathtt{R}(\tau) = \mathrm{diag}(e^{-\lambda_1 \tau}, e^{-\lambda_2 \tau}) \ , \qquad \mathtt{K}(\tau) = (0, 0) \ . \tag{4.21}$$

From now on, τ is confined as usually to an interval $[0, U)$. The functions $\rho, a, ..., e$ for this example are reported in Table 4; the length of the expressions of b, c is mainly due to the need for covering all possible values of $\lambda_1, \lambda_2, \mu$.

Comments on this example and the figures. In this case the main difficulty is the fact, following from (4.20) (4.15), that $\mathtt{J}^1(\tau)\, \mathtt{J}^2(\tau) = I_0^1 \, I_0^2 e^{-(\lambda_1 + \lambda_2)\tau}$ is small for large τ. On the other hand, $\omega(I)$ vanishes for $I^1 \, I^2 \to 0$, and in this limit many auxiliary functions diverge; so, the averaged system falls exponentially into a resonance.
In this situation one expects a rapid growth of $|\mathtt{L}|$, which is in fact confirmed by Fig.s 4a-4d; the same figures show that our estimator $\mathfrak{n}(\tau)$ approximates well the envelope of $|L(\tau/\varepsilon)|$ on $[0, U]$, when U is of the order of the unity. In Fig. 4d, a good agreement between $|\mathtt{L}(\tau/\varepsilon)|$ and $|\mathfrak{n}(\tau)|$ is attained on the longest interval among the four pictures (namely, for $\tau \in [0, 3)$). This is because we take, simultaneously, the largest value for $I_0^1 \, I_0^2$ and the lowest values for ε and $\lambda_1 + \lambda_2$.

Table 4. Auxiliary functions for Example 4.

For $I = (I^1, I^2) \in (0, +\infty)^2$, $\vartheta \in \mathbf{T}$ and $\delta I = (\delta I^1, \delta I^2) \in (-I^1, +\infty) \times (-I^2, +\infty)$:

$$s(I, \vartheta) = \frac{\mu}{2} \sin(2\vartheta) \left(-\frac{1}{I^2}, \frac{1}{I^1} \right), \qquad v(I, \vartheta) = \frac{\mu}{2 I^1 I^2} \sin^2 \vartheta \left(-\frac{1}{I^2}, \frac{1}{I^1} \right) \ ,$$

$$p(I, \vartheta) = \frac{\mu \sin(2\vartheta)}{2} \left(-\frac{\lambda_2 + \mu \cos(2\vartheta)}{I^2}, \frac{\lambda_1 + 3\mu \cos(2\vartheta)}{I^1} \right), \qquad \overline{p}(I) = (0, 0) \ ,$$

$$q(I, \vartheta) = \frac{\mu \sin^2 \vartheta}{2 I^1 I^2} \left(-\frac{2\lambda_2 + 2\mu + \lambda_1 + \mu \cos(2\vartheta)}{I^2}, \frac{\lambda_2 + 2\mu + 2\lambda_1 + 3\mu \cos(2\vartheta)}{I^1} \right),$$

$$w(I,\vartheta)=\frac{\mu\sin^2\vartheta}{2I^1I^2}\left(-\frac{\lambda_2+\mu\cos^2\vartheta}{I^2},\frac{\lambda_1+3\mu\cos^2\vartheta}{I^1}\right)\ ,\qquad u(I,\vartheta)=\frac{\mu\sin^2\vartheta}{4\,I^1I^2}\times$$
$$\times\Big(-\frac{4\lambda_2^2+6\lambda_2\mu+2\lambda_2\lambda_1+\mu\lambda_1+\mu(4\lambda_2+3\mu+\lambda_1)\cos(2\vartheta)+3\mu^2\cos^2(2\vartheta)}{I^2}\ ,$$
$$\frac{3\lambda_2\mu+2\lambda_2\lambda_1+10\mu\lambda_1+4\lambda_1^2+3\mu(\lambda_2+5\mu+4\lambda_1)\cos(2\vartheta)+15\mu^2\cos^2(2\vartheta)}{I^1}\Big)\ ,$$
$$\frac{\partial\overline{f}}{\partial I}(I)=\begin{pmatrix}-\lambda_1 & 0\\ 0 & -\lambda_2\end{pmatrix},\ \mathscr{M}(I)=\begin{pmatrix}-\lambda_1^2 & 0\\ 0 & -\lambda_2^2\end{pmatrix},\ \mathscr{G}(I,\delta I)=0,\ \mathscr{H}(I,\delta I)=0\,.$$

For $\tau\in[0,U),\ \rho(\tau):=\min(\mathtt{J}^1(\tau),\mathtt{J}^2(\tau))$.

For $\tau\in[0,U)$ and $r\in[0,J(\tau))$:

$$a(\tau,r):=\frac{|\mu|}{2}\left(\frac{1}{(\mathtt{J}^1(\tau)-r)^2}+\frac{1}{(\mathtt{J}^2(\tau)-r)^2}\right)^{1/2}\ ;$$

$$b(\tau,r):=|\mu|\ \frac{\big(b_{11}\mathtt{J}^1(\tau)^2+b_{22}\mathtt{J}^2(\tau)^2+b_1\mathtt{J}^1(\tau)r+b_2\mathtt{J}^2(\tau)r+b_0r^2\big)^{1/2}}{8(\mathtt{J}^1(\tau)-r)^2(\mathtt{J}^2(\tau)-r)^2}\ ,$$

$b_{11}:=16(\lambda_1^2+\lambda_2^2)+\lambda_1(12\lambda_2+20|\lambda_2|)+2(\lambda_1+\lambda_2)\mu+4(\lambda_1+|\lambda_2|)|\mu|+\mu^2$,
$b_{22}:=16(\lambda_1^2+\lambda_2^2)+\lambda_1(12\lambda_2+20|\lambda_2|)+6(\lambda_1+\lambda_2)\mu+12(\lambda_1+|\lambda_2|)|\mu|+9\mu^2$,
$b_1:=32(\lambda_1^2+\lambda_2^2)+64\lambda_1|\lambda_2|+12(\lambda_1+|\lambda_2|)|\mu|+2\mu^2$,
$b_2:=32(\lambda_1^2+\lambda_2^2)+64\lambda_1|\lambda_2|+36(\lambda_1+|\lambda_2|)|\mu|+18\mu^2$,
$b_0:=16(\lambda_1^2+\lambda_2^2)+\lambda_1(12\lambda_2+20|\lambda_2|)+4(\lambda_1+\lambda_2)\mu+14(\lambda_1+|\lambda_2|)|\mu|+9\mu^2$;

$$c(\tau,r):=|\mu|\frac{\big(c_{11}\mathtt{J}^1(\tau)^2+c_{22}\mathtt{J}^2(\tau)^2+c_1\mathtt{J}^1(\tau)r+c_2\mathtt{J}^2(\tau)r+c_0r^2\big)^{1/2}}{32(\mathtt{J}^1(\tau)-r)^2(\mathtt{J}^2(\tau)-r)^2}\ ,$$

$c_{11}:=1024(\lambda_1^4+\lambda_2^4)+6144\lambda_1^2\lambda_2^2+512(\lambda_1^2+\lambda_2^2)\lambda_1(3\lambda_2+5|\lambda_2|)+640(\lambda_1^3+\lambda_2^3)\mu+$
$+896(\lambda_1^3+|\lambda_2|^3)|\mu|+1920(\lambda_1+\lambda_2)\lambda_1\lambda_2\mu+2688(\lambda_1+|\lambda_2|)\lambda_1|\lambda_2||\mu|+704(\lambda_1^2+\lambda_2^2)\mu^2+$
$+32\lambda_1(17\lambda_2+27|\lambda_2|)\mu^2-24(\lambda_1+\lambda_2)\mu^3+264(\lambda_1+|\lambda_2|)|\mu|^3+27\mu^4$,
$c_{22}:=1024(\lambda_1^4+\lambda_2^4)+6144\lambda_1^2\lambda_2^2+512(\lambda_1^2+\lambda_2^2)\lambda_1(3\lambda_2+5|\lambda_2|)+384(\lambda_1^3+\lambda_2^3)\mu+$
$+1408(\lambda_1^3+|\lambda_2|^3)|\mu|+1152(\lambda_1+\lambda_2)\lambda_1\lambda_2\mu+4224(\lambda_1+|\lambda_2|)\lambda_1|\lambda_2||\mu|+2816(\lambda_1^2+\lambda_2^2)\mu^2+$
$+32\lambda_1(21\lambda_2+155|\lambda_2|)\mu^2+120(\lambda_1+\lambda_2)\mu^3+1800(\lambda_1+|\lambda_2|)|\mu|^3+675\mu^4$,
$c_1:=2048(\lambda_1^4+\lambda_2^4)+12288\lambda_1^2\lambda_2^2+8192(\lambda_1^2+\lambda_2^2)\lambda_1|\lambda_2|+3072(\lambda_1^3+|\lambda_2|^3)|\mu|+$
$+9216(\lambda_1+|\lambda_2|)\lambda_1|\lambda_2||\mu|+1408(\lambda_1^2+\lambda_2^2)\mu^2+2816\lambda_1|\lambda_2|\mu^2+576(\lambda_1+|\lambda_2|)|\mu|^3+54\mu^4$,
$c_2:=2048(\lambda_1^4+\lambda_2^4)+12288\lambda_1^2\lambda_2^2+8192(\lambda_1^2+\lambda_2^2)\lambda_1|\lambda_2|+3584(\lambda_1^3+|\lambda_2|^3)|\mu|+$
$+10752(\lambda_1+|\lambda_2|)\lambda_1|\lambda_2||\mu|+5632(\lambda_1^2+\lambda_2^2)\mu^2+11264\lambda_1|\lambda_2|\mu^2+3840(\lambda_1+|\lambda_2|)|\mu|^3+$
$1350\mu^4$,
$c_0:=1024(\lambda_1^4+\lambda_2^4)+6144\lambda_1^2\lambda_2^2+512(\lambda_1^2+\lambda_2^2)\lambda_1(3\lambda_2+5|\lambda_2|)+512(\lambda_1^3+\lambda_2^3)\mu+$
$+2048(\lambda_1^3+|\lambda_2|^3)|\mu|+1536(\lambda_1+\lambda_2)\lambda_1\lambda_2\mu+6144(\lambda_1+|\lambda_2|)\lambda_1|\lambda_2||\mu|+2816(\lambda_1^2+\lambda_2^2)\mu^2+$
$+32\lambda_1(19\lambda_2+157|\lambda_2|)\mu^2+48(\lambda_1+\lambda_2)\mu^3+1872(\lambda_1+|\lambda_2|)|\mu|^3+675\mu^4$;

$d(\tau,r):=0$; $\qquad e(\tau,r):=0$.

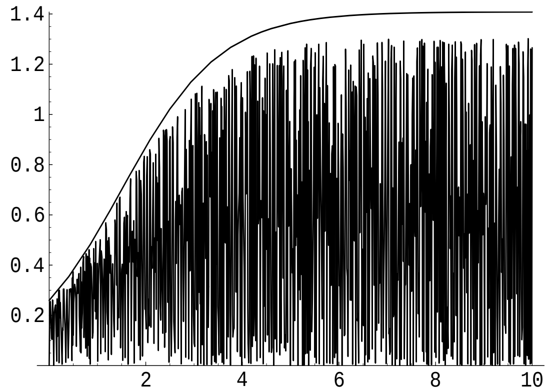

Figure 1a. $I_0 = 1/2$, $\varepsilon = 10^{-2}$, $U = 10$. Graphs of $\mathfrak{n}(\tau)$ and $|L(\tau/\varepsilon)|$. $\mathfrak{T}_{\mathfrak{N}} = 0.062s$, $\mathfrak{T}_{\mathfrak{L}} = 3.2s$.

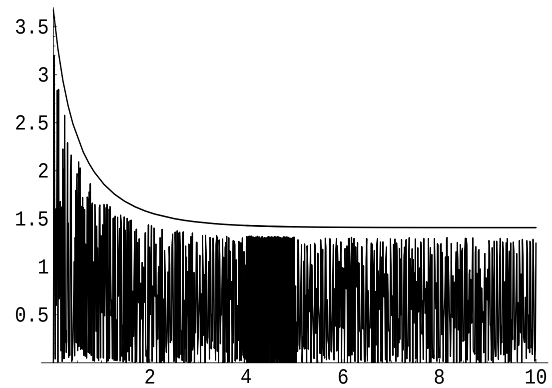

Figure 1b. $I_0 = 4$, $\varepsilon = 10^{-2}$, $U = 10$. Graphs of $\mathfrak{n}(\tau)$ and $|L(\tau/\varepsilon)|$. $\mathfrak{T}_{\mathfrak{N}} = 0.078s$, $\mathfrak{T}_{\mathfrak{L}} = 3.0s$.

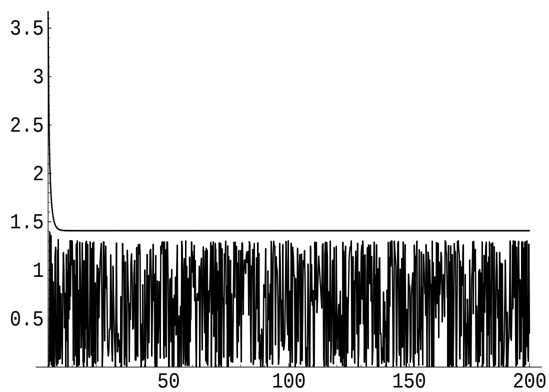

Figure 1c. $I_0 = 4$, $\varepsilon = 10^{-2}$, $U = 200$. Graphs of $\mathfrak{n}(\tau)$ and $|L(\tau/\varepsilon)|$. $\mathfrak{T}_{\mathfrak{N}} = 0.45s$, $\mathfrak{T}_{\mathfrak{L}} = 67s$.

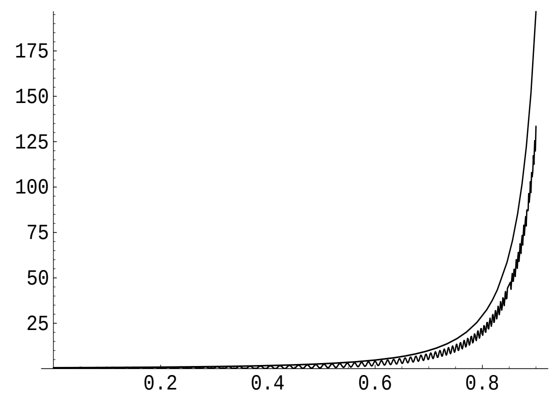

Figure 2a. $\kappa = 1, I_0 = 1, \varepsilon = 10^{-2}, U = 0.9$. Graphs of $\mathfrak{n}(\tau)$, $|L(\tau/\varepsilon)|$ (note that $\mathtt{J}(\tau) \to +\infty$ for $\tau \to 1^-$). $\mathfrak{T}_{\mathfrak{N}} = 0.032s, \mathfrak{T}_{\mathfrak{L}} = 0.36s$.

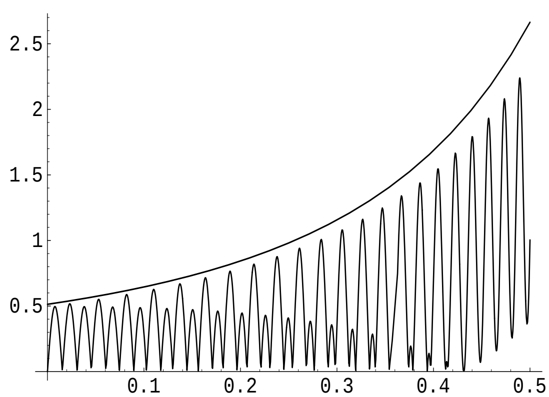

Figure 2b. $\kappa = 1, I_0 = 1, \varepsilon = 10^{-2}$ (as in Fig.2a). Graphs of $\mathfrak{n}(\tau)$, $|L(\tau/\varepsilon)|$ in a detailed view, for $\tau \in [0, 0.5]$.

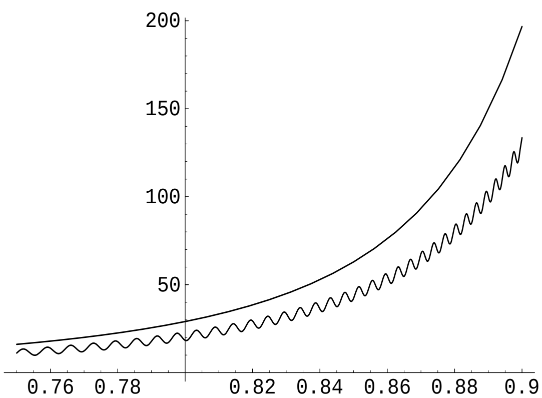

Figure 2c. $\kappa = 1, I_0 = 1, \varepsilon = 10^{-2}$ (as in Fig.2a). Graphs of $\mathfrak{n}(\tau)$, $|L(\tau/\varepsilon)|$ in a detailed view, for $\tau \in [0.75, 0.9]$.

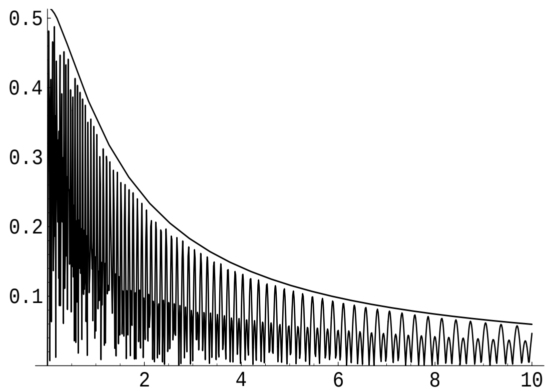

Figure 2d. $\kappa=-1, I_0=1, \varepsilon=10^{-2}, U=200$. $\mathfrak{T}_{\mathfrak{N}}=0.078s, \mathfrak{T}_{\mathfrak{L}}=0.58s$. Graphs of $\mathfrak{n}(\tau)$, $|L(\tau/\varepsilon)|$ in a detailed view, for $\tau\in[0,10]$.

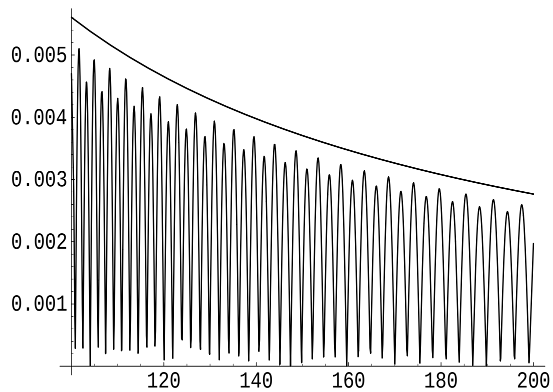

Figure 2e. $\kappa=-1, I_0=1, \varepsilon=10^{-2}$ (as in Fig.2d). Graphs of $\mathfrak{n}(\tau)$, $|L(\tau/\varepsilon)|$ in a detailed view, for $\tau\in[100,200]$.

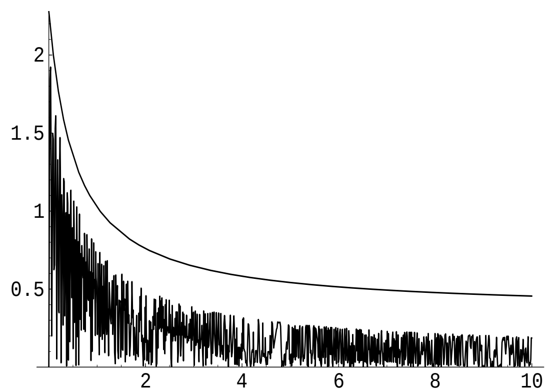

Figure 3a. $I_0=1/2$, $\varepsilon=10^{-2}$, $U=10$. Graphs of $\mathfrak{n}(\tau)$ and $|L(\tau/\varepsilon)|$. $\mathfrak{T}_{\mathfrak{N}}=0.23s$, $\mathfrak{T}_{\mathfrak{L}}=0.95s$.

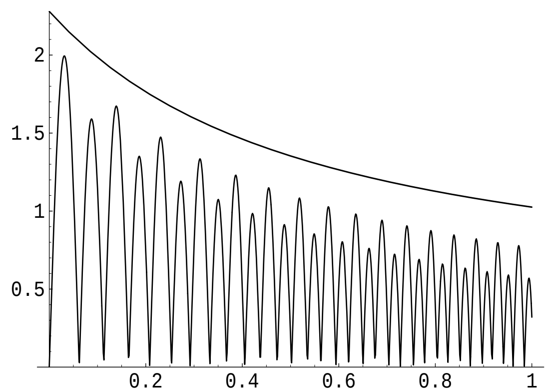

Figure 3b. $I_0=1/2$, $\varepsilon=10^{-2}$ (as in Fig.3a). Graphs of $\mathfrak{n}(\tau)$ and $|L(\tau/\varepsilon)|$ in a detailed view, for $\tau\in[0,1)$.

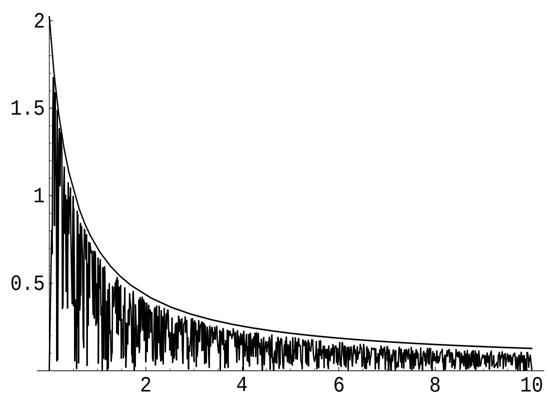

Figure 3c. $I_0=1/2$, $\varepsilon=10^{-3}$, $U=10$. Graphs of $\mathfrak{n}(\tau)$ and $|L(\tau/\varepsilon)|$. $\mathfrak{T}_{\mathfrak{N}}=0.23s$, $\mathfrak{T}_{\mathfrak{L}}=12s$.

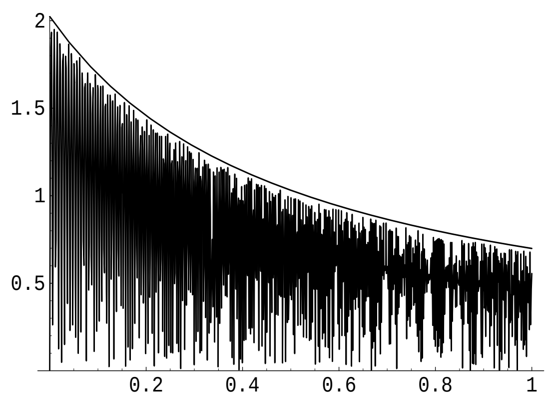

Figure 3d. $I_0=1/2$, $\varepsilon=10^{-3}$ (as in Fig.3c). Graphs of $\mathfrak{n}(\tau)$ and $|L(\tau/\varepsilon)|$ in a detailed view, for $\tau\in[0,1)$.

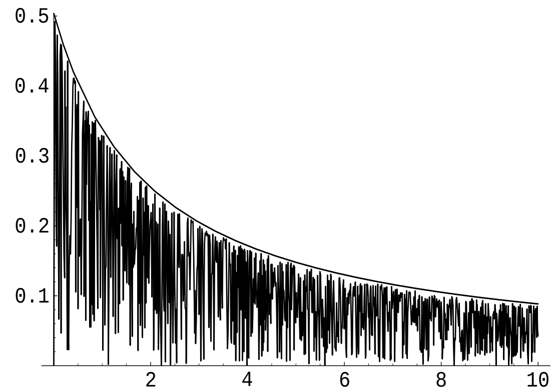

Figure 3e. $I_0 = 2$, $\varepsilon = 10^{-2}$, $U = 10$. Graphs of $\mathfrak{n}(\tau)$ and $|\mathtt{L}(\tau/\varepsilon)|$. $\mathfrak{T}_{\mathfrak{N}} = 0.16s$, $\mathfrak{T}_{\mathfrak{L}} = 1.2s$.

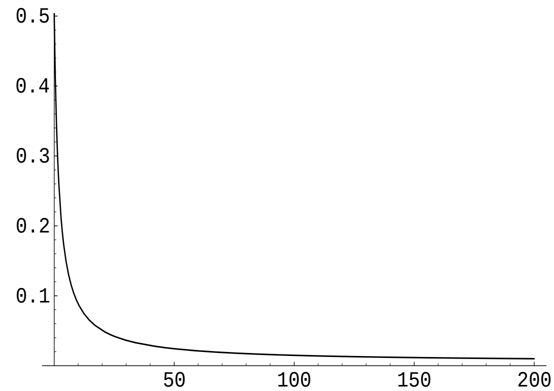

Figure 3f. $I_0 = 2$, $\varepsilon = 10^{-2}$, $U = 200$. Graph of $\mathfrak{n}(\tau)$. $\mathfrak{T}_{\mathfrak{N}} = 0.28s$, $\mathfrak{T}_{\mathfrak{L}} > 240s$.

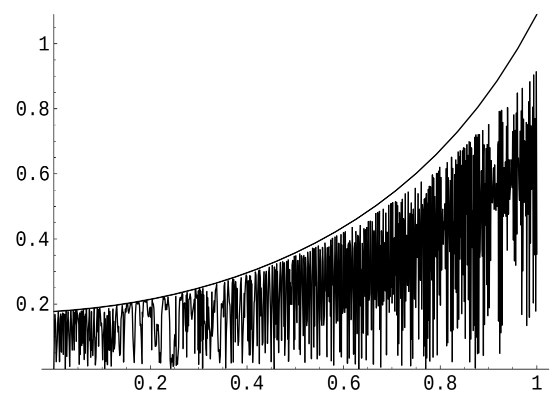

Figure 4a. $\mu = 1, \lambda_1 = 2, \lambda_2 = -1, I_0^1 = 4, I_0^2 = 4, \varepsilon = 10^{-2}, U = 1$. Graphs of $\mathfrak{n}(\tau)$ and $|L(\tau/\varepsilon)|$. $\mathfrak{T}_{\mathfrak{N}} = 0.047s, \mathfrak{T}_{\mathfrak{L}} = 1.7s$.

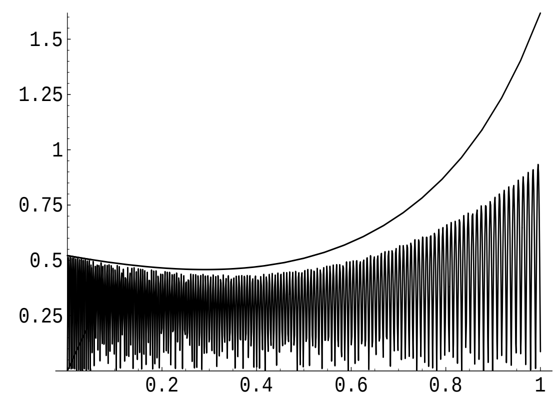

Figure 4b. $\mu = 1, \lambda_1 = 2, \lambda_2 = -1, I_0^1 = 4, I_0^2 = 1, \varepsilon = 10^{-2}, U = 1$. Graphs of $\mathfrak{n}(\tau)$ and $|L(\tau/\varepsilon)|$. $\mathfrak{T}_{\mathfrak{N}} = 0.047s, \mathfrak{T}_{\mathfrak{L}} = 0.44s$.

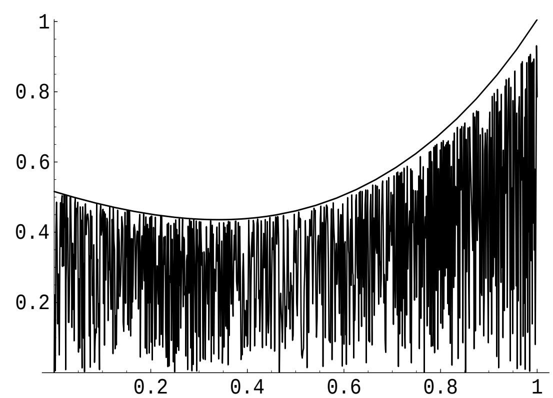

Figure 4c. $\mu = 1, \lambda_1 = 2, \lambda_2 = -1, I_0^1 = 4, I_0^2 = 1, \varepsilon = 10^{-3}, U = 1$. Graphs of $\mathfrak{n}(\tau)$ and $|L(\tau/\varepsilon)|$. $\mathfrak{T}_{\mathfrak{N}} = 0.047s, \mathfrak{T}_{\mathfrak{L}} = 4.2s$.

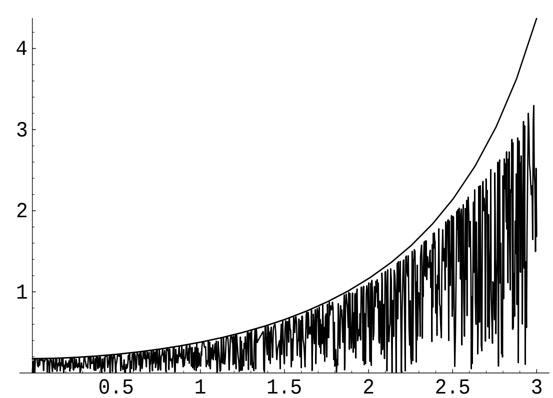

Figure 4d. $\mu = 1, \lambda_1 = 1.1, \lambda_2 = -1, I_0^1 = 4, I_0^2 = 4, \varepsilon = 10^{-3}, U = 3$. Graphs of $\mathfrak{n}(\tau)$ and $|L(\tau/\varepsilon)|$. $\mathfrak{T}_{\mathfrak{N}} = 0.046s, \mathfrak{T}_{\mathfrak{L}} = 99s$.

A Appendix. Proof of Lemma 2.1.

First of all, the Cauchy problem (2.10) has a (unique) solution on $[0,U)$, and this is C^m, because we have a linear differential equation for $\mathtt{R}$, with a C^{m-1} matrix function $\tau \mapsto \dfrac{\partial \overline{f}}{\partial I}(\mathtt{J}(\tau))$. The invertibility of $\mathtt{R}(\tau)$ follows from the Wronskian identity $\det \mathtt{R}(\tau) = \det \mathtt{R}(0) \, \exp \int_0^\tau d\tau' \, \mathrm{tr}\, \dfrac{\partial \overline{f}}{\partial I}(\mathtt{J}(\tau'))$ and from the initial condition $\mathtt{R}(0) = 1_d$; the $d = 1$ expression of $\mathtt{R}$ is obvious. The statements on $\mathtt{K}$ that follow Eq. (2.11) are also elementary (as for the C^m regularity, note that $\overline{p}(\mathtt{J})$ is a C^{m-1} function of τ). To go on, we introduce the short-hand notations

$$\mathtt{J}, \mathtt{R}, \mathtt{K}, \frac{d\mathtt{J}}{d\tau}, \text{etc.} \equiv \text{the functions } t \mapsto \mathtt{J}(\varepsilon t) \ , \ \mathtt{R}(\varepsilon t) \ , \ \mathtt{K}(\varepsilon t) \ , \ \frac{d\mathtt{J}}{d\tau}(\varepsilon t) \ , \text{etc.} \ ; \tag{A.1}$$

in the same spirit, for $h : \Lambda \times \mathbf{T} \to \mathbf{R}^d$ and $k : \Lambda \to \mathbf{R}^d$ we also intend

$$h, k, k(\mathtt{J}) \equiv \text{the functions } t \mapsto h(\mathtt{I}(t), \Theta(t)) \ , \ t \mapsto k(\mathtt{I}(t)) \ , \ t \mapsto k(\mathtt{J}(\varepsilon t)) \ . \tag{A.2}$$

In these notations, one has $\mathtt{L} = (\mathtt{I} - \mathtt{J})/\varepsilon$ and Eq.s (1.3) (1.6) imply

$$\frac{d\mathtt{L}}{dt} = \frac{1}{\varepsilon}\left(\frac{d\mathtt{I}}{dt} - \varepsilon \frac{d\mathtt{J}}{d\tau}\right) = f - \overline{f}(\mathtt{J}) \ ; \tag{A.3}$$

we continue dividing the argument in steps.
Step 1. One has

$$\frac{d\mathtt{L}}{dt} = \omega \frac{\partial s}{\partial \vartheta} + \varepsilon \frac{\partial \overline{f}}{\partial I}(\mathtt{J})\,\mathtt{L} + \frac{1}{2}\varepsilon^2 \mathscr{H}(\mathtt{J}, \varepsilon \mathtt{L})\,\mathtt{L}^2 \ . \tag{A.4}$$

In fact, Eq. (A.3) and the first Eq. (2.8) imply

$$\frac{d\mathtt{L}}{dt} = \omega \frac{\partial s}{\partial \vartheta} + \overline{f} - \overline{f}(\mathtt{J}) \ ; \tag{A.5}$$

now, it suffices to recall that $\mathtt{I} = \mathtt{J} + \varepsilon \mathtt{L}$ and use Eq. (2.19) with $(I, \delta I)$ replaced by $(\mathtt{J}, \varepsilon \mathtt{L})$.
Step 2. For each function $h \in C^1(\Lambda \times \boldsymbol{T}, \boldsymbol{R}^d)$, it is

$$\omega \frac{\partial h}{\partial \vartheta} = \frac{dh}{dt} - \varepsilon \Big(\frac{\partial h}{\partial I} f + \frac{\partial h}{\partial \vartheta} g \Big) \ . \tag{A.6}$$

This follows easily from

$$\frac{dh}{dt} = \frac{\partial h}{\partial I}\frac{d\mathtt{I}}{dt} + \frac{\partial h}{\partial \vartheta}\frac{d\Theta}{dt} = \varepsilon \frac{\partial h}{\partial I} f + \frac{\partial h}{\partial \vartheta}(\omega + \varepsilon g) \ . \tag{A.7}$$

Step 3. One has

$$\omega \frac{\partial s}{\partial \vartheta} = \frac{ds}{dt} - \varepsilon \frac{dw}{dt} - \varepsilon \overline{p}(\mathtt{J}) + \varepsilon^2 (u - \mathscr{G}(\mathtt{J}, \varepsilon \mathtt{L}) \mathtt{L}) \ . \tag{A.8}$$

To prove this, we note that Eq. (A.6) with $h = s$ and the definition (2.15) of p give

$$\omega \frac{\partial s}{\partial \vartheta} = \frac{ds}{dt} - \varepsilon p \ . \tag{A.9}$$

On the other hand, Eq.s (2.16) (A.6) with $h = w$ and the definition (2.17) of u imply

$$p = \overline{p} + \omega \frac{\partial w}{\partial \vartheta} = \overline{p} + \frac{dw}{dt} - \varepsilon u \ ; \tag{A.10}$$

furthermore, Eq. (2.18) with $(I, \delta I)$ replaced by $(\mathtt{J}, \varepsilon \mathtt{L})$ gives

$$\overline{p} = \overline{p}(\mathtt{J}) + \varepsilon \mathscr{G}(\mathtt{J}, \varepsilon \mathtt{L}) \, \mathtt{L} \ . \tag{A.11}$$

Inserting Eq. (A.11) into (A.10), and the result into (A.9), we get the equality (A.8).
Step 4. One has

$$\frac{d\mathtt{L}}{dt} - \varepsilon \frac{\partial \overline{f}}{\partial I}(\mathtt{J}) \mathtt{L} = \frac{ds}{dt} - \varepsilon \frac{dw}{dt} - \varepsilon \overline{p}(\mathtt{J}) + \varepsilon^2 (u - \mathscr{G}(\mathtt{J}, \varepsilon \mathtt{L}) \mathtt{L} + \frac{1}{2} \mathscr{H}(\mathtt{J}, \varepsilon \mathtt{L}) \mathtt{L}^2) \ . \tag{A.12}$$

This follows immediately from Eq. (A.4) and from the equality (A.8).

Introducing the next steps. Eq. (A.12) is an equality involving total derivatives, and nonderivative terms proportional to ε or ε^2. Our aim is to obtain an equality for $\mathtt{L}$ involving only total derivatives and nonderivative terms proportional to ε^2; due to the structure of the terms in ε of Eq. (A.12), this result can be achieved using the functions $\mathtt{R}$ and $\mathtt{K}$. In the sequel we will derive some identities involving $\mathtt{R}$, where the operator $\mathtt{R}(d/dt)\mathtt{R}^{-1}$ plays a major role; inserting these relations into Eq. (A.12) (and factoring out $\mathtt{R}$) we will finally obtain an identity with the desired structure, where the nonderivative terms are confined to the order ε^2.

Step 5. One has

$$\frac{d\mathtt{R}^{-1}}{dt} = -\varepsilon \mathtt{R}^{-1} \frac{\partial \overline{f}}{\partial I}(\mathtt{J}) \ . \tag{A.13}$$

For each C^1 function $\mathtt{X} : [0, U/\varepsilon) \to \boldsymbol{R}^d$, $t \mapsto \mathtt{X}(t)$, this implies

$$\frac{d\mathtt{X}}{dt} - \varepsilon \frac{\partial \overline{f}}{\partial I}(\mathtt{J}) \mathtt{X} = \mathtt{R} \frac{d}{dt}(\mathtt{R}^{-1} \mathtt{X}) \ . \tag{A.14}$$

Eq. (A.13) follows from the relation

$$0 = \frac{d}{dt}(\mathtt{R}\mathtt{R}^{-1}) = \frac{d\mathtt{R}}{dt} \mathtt{R}^{-1} + \mathtt{R} \frac{d\mathtt{R}^{-1}}{dt} = \varepsilon \frac{\partial \overline{f}}{\partial I}(\mathtt{J}) + \mathtt{R} \frac{d\mathtt{R}^{-1}}{dt} \ , \tag{A.15}$$

where, in the last passage, we have used Eq. (2.10) to express $d\mathtt{R}/dt = \varepsilon d\mathtt{R}/d\tau$. Having extablished (A.13), we consider any function $\mathtt{X}$ as above and note that

$$\frac{d\mathtt{X}}{dt} - \varepsilon \frac{\partial \overline{f}}{\partial I}(\mathtt{J})\mathtt{X} = \frac{d\mathtt{X}}{dt} + \mathtt{R}\frac{d\mathtt{R}^{-1}}{dt}\mathtt{X} = \mathtt{R}(\mathtt{R}^{-1}\frac{d\mathtt{X}}{dt} + \frac{d\mathtt{R}^{-1}}{dt}\mathtt{X}) \ , \tag{A.16}$$

whence Eq. (A.14).

Step 6. One has

$$\frac{d\mathtt{L}}{dt} - \varepsilon \frac{\partial \overline{f}}{\partial I}(\mathtt{J})\mathtt{L} = \mathtt{R}\frac{d}{dt}(\mathtt{R}^{-1}\mathtt{L}) \ , \qquad \varepsilon \frac{dw}{dt} = \varepsilon \mathtt{R}\frac{d}{dt}(\mathtt{R}^{-1}w) + \varepsilon^2 \frac{\partial \overline{f}}{\partial I}(\mathtt{J})w \ , \tag{A.17}$$

$$\varepsilon \overline{p}(\mathtt{J}) = \mathtt{R}\frac{d}{dt}(\mathtt{R}^{-1}\mathtt{K}) \tag{A.18}$$

$$\frac{ds}{dt} = \mathtt{R}\frac{d}{dt}\left(\mathtt{R}^{-1}s\right) + \varepsilon \mathtt{R}\frac{d}{dt}\left(\mathtt{R}^{-1}\frac{\partial \overline{f}}{\partial I}(\mathtt{J})v\right) - \varepsilon^2(\mathscr{M}(\mathtt{J})v + \frac{\partial \overline{f}}{\partial I}(\mathtt{J})q) \ , \tag{A.19}$$

(note that the right hand sides of Eq.s (A.17-A.19) all appear in Eq. (A.12)).
Eq.s (A.17) are mere applications of the general identity (A.14) with $\mathtt{X} = \mathtt{L}$ and $\mathtt{X} = w$ (i.e., the function $w(\mathtt{I}, \Theta)$), respectively. Eq. (A.18) follows writing (A.14) with $\mathtt{X} = \mathtt{K}$, and expressing $d\mathtt{K}/dt = \varepsilon d\mathtt{K}/d\tau$ via Eq. (2.11) . The derivation of Eq. (A.19) is a bit longer. First of all, from Eq. (A.14) with $\mathtt{X} = s$ we infer

$$\frac{ds}{dt} = \mathtt{R}\frac{d}{dt}(\mathtt{R}^{-1}s) + \varepsilon \frac{\partial \overline{f}}{\partial I}(\mathtt{J})s \ ; \tag{A.20}$$

to continue, we will reexpress $\frac{\partial \overline{f}}{\partial I}(\mathtt{J})s$ as $\mathtt{R}\times$ a total derivative, plus terms of the first order in ε. To this purpose, we write s in terms of v via Eq. (2.14), and then use Eq. (A.6) with $h = v$; this gives

$$s = \omega \frac{\partial v}{\partial \vartheta} = \frac{dv}{dt} - \varepsilon\Big(\frac{\partial v}{\partial I}f + \frac{\partial v}{\partial \vartheta}g\Big) = \frac{dv}{dt} - \varepsilon q \ , \tag{A.21}$$

the last passage following from the definition (2.15) of q. This implies

$$\frac{\partial \overline{f}}{\partial I}(\mathtt{J})s = \frac{\partial \overline{f}}{\partial I}(\mathtt{J})\frac{dv}{dt} - \varepsilon \frac{\partial \overline{f}}{\partial I}(\mathtt{J})q = \frac{d}{dt}\left(\frac{\partial \overline{f}}{\partial I}(\mathtt{J})v\right) - \frac{d}{dt}\left(\frac{\partial \overline{f}}{\partial I}(\mathtt{J})\right)v - \varepsilon \frac{\partial \overline{f}}{\partial I}(\mathtt{J})q \ . \tag{A.22}$$

On the other hand, Eq. (A.14) with $\mathtt{X} = \frac{\partial \overline{f}}{\partial I}(\mathtt{J})\,v$ and Eq. (1.6) give, respectively,

$$\frac{d}{dt}\left(\frac{\partial \overline{f}}{\partial I}(\mathtt{J})\,v\right) = \mathtt{R}\frac{d}{dt}\left(\mathtt{R}^{-1}\frac{\partial \overline{f}}{\partial I}(\mathtt{J})\,v\right) + \varepsilon\left(\frac{\partial \overline{f}}{\partial I}(\mathtt{J})\right)^2 v \ ; \tag{A.23}$$

$$\frac{d}{dt}\left(\frac{\partial\overline{f}}{\partial I}(\mathtt{J})\right) = \frac{\partial^2\overline{f}}{\partial I^2}(\mathtt{J})\,\frac{d\mathtt{J}}{dt} = \varepsilon\frac{\partial^2\overline{f}}{\partial I^2}(\mathtt{J})\,\overline{f}(\mathtt{J})\ . \tag{A.24}$$

Substituting Eq.s (A.23-A.24) into (A.22), and recalling the definition (2.17) of $\mathscr{M}$, we finally get

$$\varepsilon\frac{\partial\overline{f}}{\partial I}(\mathtt{J})s = \varepsilon\mathtt{R}\frac{d}{dt}\left(\mathtt{R}^{-1}\frac{\partial\overline{f}}{\partial I}(\mathtt{J})\,v\right) - \varepsilon^2\left(\mathscr{M}(\mathtt{J})\,v + \frac{\partial\overline{f}}{\partial I}(\mathtt{J})\,q\right)\ ; \tag{A.25}$$

inserting this result into Eq. (A.20), we obtain the desired relation (A.19).
Step 7. One has

$$\frac{d}{dt}\,(\mathtt{R}^{-1}\mathtt{L}) = \frac{d}{dt}\,(\mathtt{R}^{-1}(s-\mathtt{K})) - \varepsilon\frac{d}{dt}(\,\mathtt{R}^{-1}(w - \frac{\partial\overline{f}}{\partial I}(\mathtt{J})v)\,) + \tag{A.26}$$

$$+\varepsilon^2\,\mathtt{R}^{-1}(\,u - \frac{\partial\overline{f}}{\partial I}(\mathtt{J})(w+q) - \mathscr{M}(\mathtt{J})v - \mathscr{G}(\mathtt{J},\varepsilon\mathtt{L})\mathtt{L} + \frac{1}{2}\mathscr{H}(\mathtt{J},\varepsilon\mathtt{L})\mathtt{L}^2\,)\ .$$

To prove this, we return to (A.12) and reexpress $d\mathtt{L}/dt - \varepsilon\dfrac{\partial\overline{f}}{\partial I}(\mathtt{J})\mathtt{L}$, ds/dt, $\varepsilon dw/dt$, $\varepsilon\overline{p}(\mathtt{J})$ via Eq.s (A.17-A.18). Multiplying both sides by $\mathtt{R}^{-1}$, we obtain Eq. (A.26).
Step 8. Conclusion of the proof. We integrate Eq. (A.26) from 0 to t, explicitating the dependence of all objects on $\mathtt{I},\Theta,\mathtt{J}, t$ and taking into account the initial conditions for $\mathtt{I}$, Θ, $\mathtt{J}$, $\mathtt{R}$, $\mathtt{K}$, as well as the relations $\mathtt{L}(0) = 0$, $v(I,\vartheta_0) = w(I,\vartheta_0) = 0$. This gives an expression for $\mathtt{R}^{-1}(\varepsilon t)\,\mathtt{L}(t)$: multiplying by $\mathtt{R}(\varepsilon t)$, we get the thesis (2.13).

B Appendix. Proof of Lemma 2.3.

As anticipated, we are inspired by the proof of a similar statement in [5] (see Chapter XII, § 23, Theorem 1); therefore we merely sketch the argument. Let us define

$$\mathscr{T} := \{t_1 \in (0,T) \mid \mathfrak{l}(t) < \mathfrak{v}(t)\ \text{ for all } t\in[0,t_1)\ \}\ , \qquad T_1 := \sup\mathscr{T}\ . \tag{B.1}$$

(Note that (2.34) and (2.33) give $\mathfrak{v}(0) > \xi(0,\mathfrak{v}(0)) \geqslant 0 = \mathfrak{l}(0)$; so, by continuity, $\mathscr{T}$ is nonempty). In the sequel we will assume $T_1 < T$, and infer a contradiction.
From (B.1), it is clear that $\mathfrak{l}(T_1) \leqslant \mathfrak{v}(T_1)$. We cannot have $\mathfrak{l}(T_1) < \mathfrak{v}(T_1)$ since this, by continuity, would be against the definition of T_1; thus

$$\mathfrak{l}(T_1) = \mathfrak{v}(T_1)\ . \tag{B.2}$$

On the other hand, the assumptions of the Lemma and (B.2) imply

$$\mathfrak{l}(T_1) \leqslant_{(1)} \xi(T_1,\mathfrak{l}(T_1)) + \int_0^{T_1} dt'\eta(T_1,t',\mathfrak{l}(t')) =_{(2)} \xi(T_1,\mathfrak{v}(T_1)) + \int_0^{T_1} dt'\eta(T_1,t',\mathfrak{l}(t')) \leqslant$$

$$\leqslant_{(3)} \xi(T_1, \mathfrak{v}(T_1)) + \int_0^{T_1} dt' \eta(T_1, t', \mathfrak{v}(t')) <_{(4)} \mathfrak{v}(T_1) \ , \tag{B.3}$$

which gives again a contradiction. (For better clarity: the relations (1)(2)(3)(4) follow, respectively, from (2.33), (B.2), the monotonicity of η and (2.34)).

C Appendix. Proof of Proposition 2.5.

We begin with a Lemma; this holds under the same assumptions written at the beginning of paragraph 2E, before stating Proposition 2.4.

C.1 ***Lemma.*** *Assume that there is a family of functions $\mathfrak{n}_\delta \in C([0, U_\delta), (0, +\infty))$, labelled by a parameter $\delta \in (0, \delta_*]$, such that the following holds:*
i) $U_\delta \to U$ for $\delta \to 0^+$;
ii) for all $\delta \in (0, \delta_*]$ and $\tau \in [0, U_\delta)$, it is

$$\mathfrak{n}_\delta(\tau) < \rho(\tau)/\varepsilon \ , \tag{C.1}$$

$$\mathfrak{n}_\delta(\tau) = \delta + \alpha(\tau, \varepsilon\mathfrak{n}_\delta(\tau)) + \varepsilon|\mathtt{R}(\tau)| \int_0^\tau d\tau' \, |\mathtt{R}^{-1}(\tau')| \, \gamma(\tau', \varepsilon\mathfrak{n}_\delta(\tau'), \mathfrak{n}_\delta(\tau')) \ ; \tag{C.2}$$

iii) for each fixed $\tau \in [0, U)$, the limit $\mathfrak{n}(\tau) := \lim_{\delta \to 0^+} \mathfrak{n}_\delta(\tau)$ exists in $[0, +\infty)$ (note that $\tau \in [0, U_\delta)$ for sufficiently small δ, due to i)).
Then the solution $(\mathtt{I}, \Theta)$ of the perturbed system exists on $[0, U/\varepsilon)$ and

$$|\mathtt{L}(t)| \leqslant \mathfrak{n}(\varepsilon t) \qquad \textit{for all } t \in [0, U/\varepsilon). \tag{C.3}$$

Proof. Of course, ii) implies

$$\mathfrak{n}_\delta(\tau) > \alpha(\tau, \varepsilon\mathfrak{n}_\delta(\tau)) + \varepsilon|\mathtt{R}(\tau)| \int_0^\tau d\tau' \, |\mathtt{R}^{-1}(\tau')| \, \gamma(\tau', \varepsilon\mathfrak{n}_\delta(\tau'), \mathfrak{n}_\delta(\tau')) \tag{C.4}$$

for all $\delta \in (0, \delta_*]$ and $\tau \in [0, U_\delta)$. Therefore, Proposition 2.4 can be applied to the function $\mathfrak{n}_\delta$ on the interval $[0, U_\delta)$; this implies that $(\mathtt{I}, \Theta)$ exists on $[0, U_\delta/\varepsilon)$, and

$$|\mathtt{L}(t)| < \mathfrak{n}_\delta(\varepsilon t) \qquad \text{for all } t \in [0, U_\delta/\varepsilon). \tag{C.5}$$

Now, sending δ to zero and using iii) we easily obtain the thesis. $\diamond$

We now pass to Proposition 2.5. So, we have the assumptions at the beginning of paragraph 2E, strengthened by the smoothness requirements (2.45) for a, b, c, d, e .

Proof of Proposition 2.5. For the sake of brevity, we put

$$\alpha_0 : \Sigma \to \mathbf{R}, \qquad \ell \mapsto \alpha_0(\ell) := \alpha(0, \varepsilon\ell) \tag{C.6}$$

and extend this definition to any $\delta \geqslant 0$ setting

$$\alpha_\delta : \Sigma \to \mathbf{R} \;, \qquad \ell \mapsto \alpha_\delta(\ell) := \alpha_0(\ell) + \delta \;. \tag{C.7}$$

We proceed in several steps.
Step 1. For each $\delta \geqslant 0$, α_δ is a contractive map. In fact, for all $\ell, \ell' \in \Sigma$, we have

$$|\alpha_\delta(\ell) - \alpha_\delta(\ell')| = \varepsilon \left| \frac{\partial \alpha}{\partial r}(0, \varepsilon \ell) \right| |\ell - \ell'| \leqslant \varepsilon M |\ell - \ell'| \;. \tag{C.8}$$

But $\varepsilon M < 1$ by the first inequality (2.47), so the thesis is proved.
Step 2. There is $\delta_* > 0$ such that, for all $\delta \in [0, \delta_*]$, α_δ sends Σ into itself. In fact, for any $\delta \geqslant 0$ and $\ell \in \Sigma$,

$$\begin{aligned} |\alpha_\delta(\ell) - \ell_*| &= |\alpha_0(\ell) + \delta - \ell_*| \leqslant |\alpha_0(\ell) - \alpha_0(\ell_*)| + |\alpha_0(\ell_*) - \ell_*| + \delta \leqslant \\ &\leqslant \varepsilon M |\ell - \ell_*| + |\alpha_0(\ell_*) - \ell_*| + \delta \leqslant \varepsilon M \sigma + |\alpha_0(\ell_*) - \ell_*| + \delta \;, \end{aligned} \tag{C.9}$$

where the second inequality follows from Eq. (C.8) with $\delta = 0$. Now, let us define

$$\delta_* := (1 - \varepsilon M)\sigma - |\alpha_0(\ell_*) - \ell_*| \;, \tag{C.10}$$

and note that $\delta_* > 0$ by (2.48). For $\delta \in [0, \delta_*]$ and $\ell \in \Sigma$, Eq.s (C.9), (C.10) imply $|\alpha_\delta(\ell) - \ell_*| \leqslant \sigma$, i.e., $\alpha_\delta(\ell) \in \Sigma$.
Step 3. For all $\delta \in [0, \delta_*]$, the map α_δ has a unique fixed point $\ell_\delta \in \Sigma$, which depends continuously on δ. Existence and uniqueness of the fixed point follows from the Banach theorem on contractions; to prove continuity we note that, for all $\delta, \delta' \in [0, \delta_*]$,

$$\begin{aligned} |\ell_\delta - \ell_{\delta'}| &= |\alpha_\delta(\ell_\delta) - \alpha_{\delta'}(\ell_{\delta'})| = |\alpha_0(\ell_\delta) + \delta - \alpha_0(\ell_{\delta'}) - \delta'| \leqslant \\ &\leqslant |\alpha_0(\ell_\delta) - \alpha_0(\ell_{\delta'})| + |\delta - \delta'| \leqslant \varepsilon M |\ell_\delta - \ell_{\delta'}| + |\delta - \delta'| \;, \end{aligned} \tag{C.11}$$

the last inequality depending on (C.8) with $\delta = 0$. This implies

$$|\ell_\delta - \ell_{\delta'}| \leqslant \frac{|\delta - \delta'|}{1 - \varepsilon M} \;; \tag{C.12}$$

so the map $\delta \mapsto \ell_\delta$ is Lipschitz, and a fortiori continuous.
Step 4. Proving the thesis of i). This follows from Step 3, with $\delta = 0$.
Step 5. Proving the thesis of ii). For any $\delta \in [0, \delta_*]$, let ℓ_δ be as in Step 3. From the standard continuity theorems for the solutions of a parameter-dependent Cauchy problem, we know that there is a family $(U_\delta, \mathfrak{m}_\delta, \mathfrak{n}_\delta)_{\delta \in (0, \delta_*]}$ with the following properties a) b):

a) for all $\delta \in (0, \delta_*]$, it is $\mathfrak{m}_\delta, \mathfrak{n}_\delta \in C^1([0, U_\delta), \mathbf{R})$; furthermore, these functions fulfil the equations

$$\frac{d\mathfrak{m}_\delta}{d\tau} = |\mathtt{R}^{-1}|\gamma(\cdot, \varepsilon\mathfrak{n}_\delta, \mathfrak{n}_\delta) \ , \qquad \mathfrak{m}_\delta(0) = 0 \ , \tag{C.13}$$

$$\frac{d\mathfrak{n}_\delta}{d\tau} = \Big(1 - \varepsilon \frac{\partial \alpha}{\partial r}(\cdot, \varepsilon\mathfrak{n}_\delta)\Big)^{-1} \left(\frac{\partial \alpha}{\partial \tau}(\cdot, \varepsilon\mathfrak{n}_\delta) + \varepsilon|\mathtt{R}||\mathtt{R}^{-1}|\,\gamma(\cdot, \varepsilon\mathfrak{n}_\delta, \mathfrak{n}_\delta) + \varepsilon|\mathtt{R}|^{-1}\,\Big(\mathtt{R}\bullet\frac{d\mathtt{R}}{d\tau}\Big)\mathfrak{m}_\delta\right),$$
$$\mathfrak{n}_\delta(0) = \ell_\delta \tag{C.14}$$

with the domain conditions

$$0 < \mathfrak{n}_\delta < \rho/\varepsilon \ , \qquad \frac{\partial \alpha}{\partial r}(\cdot, \varepsilon\mathfrak{n}_\delta) < 1/\varepsilon \ . \tag{C.15}$$

b) One has

$$U_\delta \to_{\delta\to 0^+} U, \quad \mathfrak{n}_\delta(t) \to_{\delta\to 0^+} \mathfrak{n}(t), \quad \mathfrak{m}_\delta(t) \to_{\delta\to 0^+} \mathfrak{m}(t) \ \text{ for all } t \in [0, U), \tag{C.16}$$

where $\mathfrak{m}$, $\mathfrak{n}$ are as stated in ii).
Let us consider the pair $\mathfrak{m}_\delta, \mathfrak{n}_\delta$ for any $\delta \in (0, \delta_*]$. Then, integrating (C.13),

$$\mathfrak{m}_\delta(\tau) = \int_0^\tau d\tau'\,|\mathtt{R}^{-1}(\tau')\,|\gamma(\tau', \varepsilon\mathfrak{n}_\delta(\tau'), \mathfrak{n}_\delta(\tau')) \qquad \text{for } \tau \in [0, U_\delta). \tag{C.17}$$

Furthermore, from Eq. (C.14) we infer

$$0 = \Big(1 - \varepsilon\frac{\partial \alpha}{\partial r}(\cdot, \varepsilon\mathfrak{n}_\delta)\Big)\frac{d\mathfrak{n}_\delta}{d\tau} - \left(\frac{\partial \alpha}{\partial \tau}(\cdot, \varepsilon\mathfrak{n}_\delta) + \varepsilon|\mathtt{R}||\mathtt{R}^{-1}|\,\gamma(\cdot, \varepsilon\mathfrak{n}_\delta, \mathfrak{n}_\delta) + \varepsilon|\mathtt{R}|^{-1}\,\Big(\mathtt{R}\bullet\frac{d\mathtt{R}}{d\tau}\Big)\,\mathfrak{m}_\delta\right) =$$

$$= \Big(1 - \varepsilon\frac{\partial \alpha}{\partial r}(\cdot, \varepsilon\mathfrak{n}_\delta)\Big)\frac{d\mathfrak{n}_\delta}{d\tau} - \left(\frac{\partial \alpha}{\partial \tau}(\cdot, \varepsilon\mathfrak{n}_\delta) + \varepsilon|\mathtt{R}|\frac{d\mathfrak{m}_\delta}{d\tau} + \varepsilon\frac{d|\mathtt{R}|}{d\tau}\mathfrak{m}_\delta\right) \ ; \tag{C.18}$$

the last passage depends on Eq. (C.13) for $\mathfrak{m}_\delta$, and from the identity $d|\mathtt{R}|/d\tau = d\sqrt{\mathtt{R}\bullet\mathtt{R}}/d\tau = |\mathtt{R}|^{-1}(\mathtt{R}\bullet d\mathtt{R}/d\tau)$. The result (C.18) can be rephrased as

$$0 = \frac{d}{d\tau}(\mathfrak{n}_\delta - \alpha(\cdot, \varepsilon\mathfrak{n}_\delta) - \varepsilon|\mathtt{R}|\mathfrak{m}_\delta) \ ; \tag{C.19}$$

the constant value of the above function can be computed setting $\tau = 0$, and is

$$\mathfrak{n}_\delta(0) - \alpha(0, \varepsilon\mathfrak{n}_\delta(0)) = \ell_\delta - \alpha(0, \varepsilon\ell_\delta) = \ell_\delta - \alpha_0(\ell_\delta) = \delta \tag{C.20}$$

(recall the initial condition in (C.14), Eq.s (C.6) (C.7) and Step 3, giving $\ell_\delta = \alpha_\delta(\ell_\delta) = \alpha_0(\ell_\delta) + \delta$). Therefore,

$$\mathfrak{n}_\delta(\tau) - \alpha(\tau, \varepsilon\mathfrak{n}_\delta(\tau)) - \varepsilon|\mathtt{R}(\tau)|\,\mathfrak{m}_\delta(\tau) = \delta \qquad \text{for } \tau \in [0, U_\delta) \ . \tag{C.21}$$

From Eq.s (C.21) and (C.17), we see that $\mathfrak{n}_\delta$ fulfils Eq. (C.2) of Lemma C.1. Due to Eq. (C.16) on the limit for $\delta \to 0^+$, from Lemma C.1 we finally obtain the thesis. $\diamond$

D Appendix. The functions a of the examples.

Example 1. One must determine a function fulfilling Eq. (2.23) for $\tau \in [0, U)$, $\delta J \in (-\mathtt{J}(\tau), \mathtt{J}(\tau))$ and $\vartheta \in \mathbf{T}$. Neither $\mathtt{K}$ nor $\mathtt{R}$ (nor the initial datum) play a significant role in this computation, since $\mathtt{K} = 0$, $s(I_0, \vartheta_0) = 0$ and $\mathtt{R}(\tau)$ appears in Eq. (2.23) as a multiplier for the second of these vanishing terms. In conclusion, to obtain a we can simply bind $|s(\mathtt{J}(\tau) + \delta J, \vartheta)|$ in terms of $\mathtt{J}(\tau)$ and $r := |\delta J|$.
Consider any point $I \in \Lambda$; of course,

$$\max_{\vartheta \in \mathbf{T}} |s(I, \vartheta)| = \left(\max_{\vartheta \in \mathbf{T}} s^2(I, \vartheta) \right)^{1/2} . \tag{D.1}$$

Derivating with respect to ϑ, one finds that the maximum of s^2 is attained for $\cos^2 \vartheta = 1/2 + (1 - \sqrt{1 + 2I^2})/(4I)$; by elementary computations, this gives

$$\max_{\vartheta \in \mathbf{T}} |s(I, \vartheta)| = \mathrm{a}(I) \ , \tag{D.2}$$

where a is the C^∞, strictly increasing function given by

$$\mathrm{a} : (0, +\infty) \to (0, +\infty) \ , I \mapsto \mathrm{a}(I) := \frac{1}{8} \big(-2 + 10I^2 + I^4 + 2(1 + 2I^2)^{3/2} \big)^{1/2} \ . \tag{D.3}$$

Let $\tau \in [0, U)$, $\delta J \in (-\mathtt{J}(\tau), \mathtt{J}(\tau))$, $\vartheta \in \mathbf{T}$ and $r := |\delta J|$. Then,

$$|s(\mathtt{J}(\tau) + \delta J, \vartheta)| \leqslant \mathrm{a}(\mathtt{J}(\tau) + \delta J) \leqslant \mathrm{a}(\mathtt{J}(\tau) + r) \ ; \tag{D.4}$$

the last term above is just the function $a(\tau, r)$ of Table 1.
Example 2. We refer again to Eq. (2.23); as in the previous example, $\mathtt{R}(\tau)$ plays no role, since it appears in Eq. (2.24) as a multiplier for the term $s(I_0, \vartheta_0) = 0$. A simple computation gives

$$|s(\mathtt{J}(\tau) + \delta J, \vartheta) - \mathtt{K}(\tau)| = | -\frac{\kappa}{2} (\mathtt{J}(\tau) + \delta J) \sin(2\vartheta) - \mathtt{K}(\tau)| \leqslant \tag{D.5}$$

$$\leqslant \frac{1}{2} (\mathtt{J}(\tau) + |\delta J|) + |\mathtt{K}(\tau)| = \frac{1}{2} (\mathtt{J}(\tau) + |\delta J|) - \mathtt{K}(\tau) \ ;$$

this means that Eq. (2.23) is fulfilled by the function a in the Table 2.
Example 3. Again, $\mathtt{R}$ and $\mathtt{K}$ play no role in the analysis of Eq. (2.23) and it suffices to bind $|s(\mathtt{J}(\tau) + \delta J, \vartheta)|$ in terms of $r := |\delta J|$. Clearly,

$$|s(\mathtt{J}(\tau) + \delta J, \vartheta)| \leqslant \frac{1}{|\mathtt{J}(\tau) + \delta J|} \leqslant \frac{1}{\mathtt{J}(\tau) - r} \ ; \tag{D.6}$$

therefore, Eq. (2.23) is fulfilled by the function of Table 3.
Example 4. In the left hand side of Eq. (2.23), the terms $\mathtt{K}(\tau)$ and $s(I_0, \vartheta_0)$ are zero; so, to find a we must bind $|s(\mathtt{J}(\tau) + \delta J, \vartheta)|$ in terms of $r := |\delta J|$. Maximization with respect to ϑ can be done analytically; as a final result, Eq. (2.23) is fulfilled by the function a in Table 4.
In each example, the function a determined as above gives an accurate bound on the left hand side of Eq. (2.23).

E Appendix. The functions $\boldsymbol{b}, \boldsymbol{c}$ of the examples.

i) In comparison with a, the functions b, c, d, e in Eq.s (2.24–2.27) can be constructed using rougher majorizations, see the comments in paragraph 3A. Here we fix the attention on b and c, since the functions d, e of the examples are obtained trivially.
ii) In all the examples, to find b and c we must essentially derive a majorant for an expressions of the form $h(\mathtt{J}(\tau), \delta J, \vartheta)$, where $h(J, \delta J, \vartheta)$ is a trigonometric polynomial in ϑ, whose coefficients are polynomials in J and δJ. The majorant should depend only on $\mathtt{J}(\tau)$ and $|\delta J|$; so, the problem is reduced to finding a function k such that

$$h(J, \delta J, \vartheta) \leqslant k(J, r) \qquad \text{for } \vartheta \in \mathbf{T} \text{ and } r := |\delta J| \ . \tag{E.1}$$

Let us exemplify this situation in the construction of b; computations for c are quite similar.

Example 1. To find b we can bind $\big(w(\mathtt{J}(\tau) + \delta J, \vartheta) - \dfrac{\partial \overline{f}}{\partial I}(\mathtt{J}(\tau))\, v(\mathtt{J}(\tau) + \delta J, \vartheta)\big)^2$, which has the form $h(\mathtt{J}(\tau), \delta J, \vartheta)$ with h a polynomial as above; the square root of the majorant $k(\mathtt{J}(\tau), r)$ is $b(\tau, r)$.
Example 2. This computation is very similar to the one for Example 1.
Example 3. The left hand side of Eq. (2.24) is

$$|w(\mathtt{J}(\tau) + \delta J, \vartheta) - \frac{\partial \overline{f}}{\partial I}(\mathtt{J}(\tau)) v(\mathtt{J}(\tau) + \delta J, \vartheta)| = \tag{E.2}$$

$$= \frac{3 - 4\cos\vartheta + \cos(2\vartheta)}{4(\mathtt{J}(\tau) + \delta J)^3} \leqslant \frac{2}{(\mathtt{J}(\tau) + \delta J)^3} \leqslant \left.\frac{2}{(\mathtt{J}(\tau) - r)^3}\right|_{r = |\delta J|} .$$

Example 4. In this case,

$$|w(\mathtt{J}(\tau) + \delta J, \vartheta) - \frac{\partial \overline{f}}{\partial I}(\mathtt{J}(\tau))\, v(\mathtt{J}(\tau) + \delta J, \vartheta)|^2 = \frac{h(\mathtt{J}(\tau), \delta J, \vartheta)}{(\mathtt{J}^1(\tau) + \delta J^1)^4 (\mathtt{J}^2(\tau) + \delta J^2)^4} \tag{E.3}$$

with h a polynomial as before. After finding for h a bound of the form (E.1), we combine it with the obvious relation $(\mathtt{J}^i(\tau) + \delta J^i)^{-4} \leqslant (\mathtt{J}^i(\tau) - r)^{-4}$ for $r := |\delta J|$; the square root of the final majorant is $b(\tau, r)$.
iii) Up to now, we have not explained how to get elementary bounds of the form (E.1) on a polynomial h. Here we illustrate a general procedure (computations to apply it in Examples 1-4 are generally too tedious to be made by hand, but are easily implemented on MATHEMATICA).
iii a) In the expression of $h(J, \delta J, \vartheta)$, if $d = 1$ we put $\delta J = r\cos\psi$ with $\psi = 0$ or π; if $d = 2$, we set $\delta J = (r\cos\psi, r\sin\psi)$ with $\psi \in \mathbf{T}$.
iii b) Now, $h(J, \delta J, \vartheta)$ has the form of a trigonometric polynomial in ϑ, ψ with coefficients depending on J. We write this in a canonical form, reexpressing any term in ϑ and ψ as a linear combination of sines and cosines (e.g., $\cos^4\vartheta \sin(2\vartheta)^2 = (1/32)(5 + 4\cos(2\vartheta) - 4\cos(4\vartheta) - 4\cos(6\vartheta) - \cos(8\vartheta))$.
iii c) As a final step, we bind each summand of h using the relations $|\cos|, |\sin| \leqslant 1$.

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References

[1] V.I. Arnold, *Mathematical methods of classical mechanics*, Springer-Verlag, New York (1978).

[2] V.I. Arnold, *Geometrical methods in the theory of ordinary differential equations*, Springer-Verlag, New York (1983).

[3] N.N. Bogoliubov, Y.A. Mitropolski, *Asymptotic methods in the theory of non-linear oscillations*, Gordon and Breach, New York (1961).

[4] P. Lochak, C. Meunier, *Multiphase averaging for classical systems*, Springer-Verlag, New York (1988).

[5] D.S. Mitrinovic, J.E. Pecaric, A.M. Fink, *Inequalities involving functions and their integrals and derivatives*, Kluwer, Dordrecht (1991).

[6] C.Morosi, L.Pizzocchero, *On approximate solutions of semilinear evolution equations*, Rev.Math.Phys. **16**, 383-420 (2004).

[7] J.A. Sanders, F. Verhulst, *Averaging methods in nonlinear dynamical systems*, Springer-Verlag, New York (1985).

[8] D.R. Smith, *Singular perturbation theory*, Cambridge Univ. Press, Cambridge (1985).

[9] E. Zeidler, *Non linear functional analysis and its applications II/B. Nonlinear monotone operators*, Springer-Verlag, New York (1990).